

# FLAT VECTOR BUNDLES AND ANALYTIC TORSION ON ORBIFOLDS

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**ABSTRACT.** This article is devoted to a study of flat orbifold vector bundles. We construct a bijection between the isomorphic classes of proper flat orbifold vector bundles and the equivalence classes of representations of the orbifold fundamental groups of base orbifolds.

We establish a Bismut-Zhang like anomaly formula for the Ray-Singer metric on the determine line of the cohomology of a compact orbifold with coefficients in an orbifold flat vector bundle.

We show that the analytic torsion of an acyclic unitary flat orbifold vector bundle is equal to the value at zero of a dynamical zeta function when the underlying orbifold is a compact locally symmetric space of the reductive type, which extends one of the results obtained by the first author for compact locally symmetric manifolds.

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## INTRODUCTION

Orbifolds were introduced by Satake [Sa56] under name of  $V$ -manifold as manifolds with quotient singularities. They appear naturally, for example, in the geometry of 3-manifolds, in the symplectic reduction, in the problems on moduli spaces, and in string theory, etc.

It is natural to consider the index theoretic problem and the associated secondary invariants on orbifolds. Satake [Sa57] and Kawasaki [K78, K79] extended the classical Gauss-Bonnet-Chern Theorem, the Hirzebruch signature Theorem and the Riemann-Roch-Hirzebruch Theorem. For the secondary invariants, Ma [Ma05] studied the holomorphic torsions and Quillen metrics associated with holomorphic orbifold vector bundles, and Farsi [Fa07] introduced an orbifold version eta invariant and extended the Atiyah-Patodi-Singer Theorem. In this article, we study flat orbifold vector bundles and the associated secondary invariants, i.e., analytic torsions or more precisely Ray-Singer metrics.

Let us recall some results on flat vector bundles on manifolds. Let  $Z$  be a connected smooth manifold, and let  $F$  be a complex flat vector bundle on  $Z$ . Equivalently,  $F$  can be obtained via a complex representation of the fundamental group  $\pi_1(Z)$  of  $Z$ , which is called the holonomy representation. Denote by  $H^*(Z, F)$  the cohomology of the sheaf of locally constant sections of  $F$ .

Assume that  $Z$  is compact. Given metrics  $g^{TZ}$  and  $g^F$  on  $TZ$  and  $F$ , the Ray-Singer metric [RS71] on the determinant line  $\lambda$  of  $H^*(Z, F)$  is defined by the product of the analytic torsion with the  $L^2$ -metric on  $\lambda$  obtained via Hodge Theory.

If  $g^F$  is flat, or equivalently if the holonomy representation is unitary, then the celebrated Cheeger-Müller Theorem [C79, M78] tells us that in this case the Ray-Singer metric coincides with the so-called Reidemeister metric [Re36], which is a topological invariant of the unitarily flat vector bundles constructed with the help of a triangulation on  $Z$ . Bismut-Zhang [BZ92] and Müller [M93] simultaneously considered generalizations of this result. In [M93], Müller extended it to the case where  $g^F$  is unimodular or equivalently the holonomy representation is unimodular. In [BZ92], Bismut and Zhang studied the dependence of the Ray-Singer metric on  $g^{TZ}$  and  $g^F$ . They gave an anomaly formula [BZ92, Theorem 0.1] for the variation of the logarithm of the Ray-Singer metric on  $g^{TZ}$  and  $g^F$  as an integral of a locally calculable Chern-Simons form on  $Z$ . They

generalized the original Cheeger-Müller Theorem to arbitrary flat vector bundles with arbitrary Hermitian metrics [BZ92, Theorem 0.2]. In [BZ94], Bismut and Zhang also considered the extensions to the equivariant case. Note that both in [BZ92, BZ94], the existence of a Morse function whose gradient satisfies the Smale transversality condition [Sm61, Sm67] plays an important role.

From the dynamical side, motivated by a remarkable similarity [M68, Section 3] between the analytic torsion and Weil's zeta function, Fried [F86] showed that, when the underlying manifold is hyperbolic, the analytic torsion of an acyclic unitarily flat vector bundle is equal to the value at zero of the Ruelle dynamical zeta function. In [F87, p.66 Conjecture], he conjectured similar results hold true for more general spaces.

In [Sh16a, Sh16b], following the early contribution of Fried [F86] and Moscovici-Stanton [MoSt91], the author showed the Fried conjecture on compact locally symmetric manifolds of the reductive type. The proof is based on Bismut's explicit semisimple orbital integral formula [B11, Theorem 6.1.1].

In this article, we extend most of the above results to orbifolds. The case of Cheeger-Müller Theorem involves, however, real difficulties as the existence of a Morse function whose gradient satisfies the Smale transversality condition is not clear to us, and is therefore not considered here.

Now, we will describe our results in more details, and explain the techniques used in their proof.

**0.1. Orbifold fundamental group and holonomy representation.** Let  $Z$  be a connected orbifold with the associated groupoid  $\mathcal{G}$ . Following Thurston [T80], let  $X$  be the universal covering orbifold of  $Z$  with the deck transformation group  $\Gamma$ , which is called orbifold fundamental group of  $Z$ . Then,  $Z = \Gamma \backslash X$ . In an analogous way as in the classical homotopy theory of ordinary paths on topological spaces, Haefliger [H90] introduced the  $\mathcal{G}$ -paths and their homotopy theory. He gave an explicit construction of  $X$  and  $\Gamma$  following the classical methods.

If  $F$  is a complex proper flat orbifold vector bundle of rank  $r$ , in Section 2, we constructed a parallel transport along a  $\mathcal{G}$ -path. In this way, we obtain a representation  $\rho : \Gamma \rightarrow \mathrm{GL}_r(\mathbb{C})$  of  $\Gamma$ , which is called the holonomy representation of  $F$ . Denote by  $\mathcal{M}_r^{\mathrm{pr}}(Z)$  the isomorphic classes of complex proper flat orbifold vector bundles of rank  $r$  on  $Z$ , and denote by  $\mathrm{Hom}(\Gamma, \mathrm{GL}_r(\mathbb{C}))/\sim$  the equivalence classes of complex representations of  $\Gamma$  of dimension  $r$ . We show the following theorem.

**Theorem 0.1.** *The above construction descends to a well-defined bijection*

$$(0.1) \quad \mathcal{M}_r^{\mathrm{pr}}(Z) \rightarrow \mathrm{Hom}(\Gamma, \mathrm{GL}_r(\mathbb{C}))/\sim.$$

The difficulty of the proof lies in the injectivity, which consists in showing that  $F$  is isomorphic to the quotient of  $X \times \mathbb{C}^r$  by the  $\Gamma$ -action induced by the deck transformation on  $X$  and by the holonomy representation on  $\mathbb{C}^r$ . Indeed, applying Haefliger's construction, in subsection 2.5, we show directly that the universal covering orbifold of the total space of  $F$  is  $X \times \mathbb{C}^r$ . Moreover, its deck transformation group is isomorphic to  $\Gamma$  with the desired action on  $X \times \mathbb{C}^r$ .

We remark that on the universal covering orbifold there exist non trivial and non proper flat orbifold vector bundles. Thus, Theorem 0.1 no longer holds true for non proper orbifold vector bundles.

On the other hand, for a general orbifold vector bundle  $E$  which is not necessarily proper, there exists a proper subbundle  $E^{\text{pr}}$  of  $E$  such that

$$(0.2) \quad C^\infty(Z, E) = C^\infty(Z, E^{\text{pr}}).$$

Moreover, if  $E$  is flat,  $E^{\text{pr}}$  is also flat. For a  $\Gamma$ -space  $V$ , we denote by  $V^\Gamma$  the set of fixed points in  $V$ . By Theorem 0.1 and (0.2), we get:

**Corollary 0.2.** *For any (possibly non proper) flat orbifold vector bundle  $F$  on a connected orbifold  $Z$ , there exists a representation of the orbifold fundamental group  $\rho : \Gamma \rightarrow \text{GL}_r(\mathbb{C})$  such that*

$$(0.3) \quad C^\infty(Z, F) = C^\infty(X, \mathbb{C}^r)^\Gamma.$$

By abuse of notation, in this case, although  $\rho$  is not unique, we still call  $\rho$  a holonomy representation of  $F$ .

**0.2. Analytic torsion on orbifolds.** Assume  $Z$  is a compact orbifold of dimension  $m$ . Let  $\Sigma Z$  be the strata of  $Z$ , which has a natural orbifold structure. Write  $Z \coprod \Sigma Z = \coprod_{i=0}^{l_0} Z_i$  as a disjoint union of connected components. We denote  $m_i \in \mathbb{N}$  the multiplicity of  $Z_i$  (c.f. (2.18)). Let  $F$  be a complex flat orbifold vector bundle on  $Z$ . Let  $\lambda$  be the determinant of the cohomology  $H^\bullet(Z, F)$  (c.f. (4.41)).

Let  $g^{TZ}$  and  $g^F$  be metrics on  $TZ$  and  $F$ . Denote by  $\square^Z$  the associated Hodge Laplacian acting on the space  $\Omega^\bullet(Z, F)$  of smooth forms with values in  $F$ . By the orbifold Hodge Theorem [DY16, Proposition 2.1], we have the canonical isomorphism

$$(0.4) \quad H^\bullet(Z, F) \simeq \ker \square^Z.$$

As in the case of smooth manifolds, by the short time asymptotic expansions of the heat trace (c.f. [Ma05, Proposition 2.1]), the analytic torsion  $T_Z(F)$  is still well-defined. It is a real positive number defined by the following weighted product of the zeta regularized determinants

$$(0.5) \quad T(F) = \prod_{i=1}^m \det(\square^Z|_{\Omega^i(Z, F)})^{(-1)^i i/2}.$$

Let  $|\cdot|_\lambda^{\text{RS}, 2}$  be the  $L^2$ -metric on  $\lambda$  induced by  $g^{TZ}, g^F$  via (0.4). The Ray-Singer metric is then given by

$$(0.6) \quad \|\cdot\|_\lambda^{\text{RS}} = T(F) |\cdot|_\lambda^{\text{RS}}.$$

We remark that as in the smooth case, if  $Z$  is of even dimension and orientable, in Proposition 4.7, we show that  $T(F) = 1$ .

In Section 4, we study the dependance of  $\|\cdot\|_\lambda^{\text{RS}, 2}$  on  $g^{TZ}$  and  $g^F$ . To state our result, let us introduce some notations. Let  $(g'^{TZ}, g'^F)$  be another pair of metrics. Let  $\|\cdot\|_\lambda'^{\text{RS}, 2}$  be the Ray-Singer metric for  $(g'^{TZ}, g'^F)$ . Let  $\nabla^{TZ}$  and  $\nabla'^{TZ}$  be the respective Levi-Civita connections on  $TZ$  for  $g^{TZ}$  and  $g'^{TZ}$ . Denote by  $o(TZ)$  be the orientation line of  $Z$ . Consider the Euler form  $e(TZ, \nabla^{TZ}) \in \Omega^m(Z, o(TZ))$  and the first odd Chern form  $\frac{1}{2}\theta(\nabla^F, g^F) = \frac{1}{2}\text{Tr}[g^{F, -1}\nabla^F g^F] \in \Omega^1(Z)$ . Denote by  $e(Z_i, \nabla^{TZ_i}) \in \Omega^{\dim Z_i}(Z_i, o(TZ_i))$  and  $\theta_i(\nabla^F, g^F) \in \Omega^1(Z_i)$  the canonical extensions of  $e(TZ, \nabla^{TZ})$  and  $\theta(\nabla^F, g^F)$  to  $Z_i$  (c.f. subsection 3.3). Let  $\tilde{e}(TZ_i, \nabla^{TZ_i}, \nabla'^{TZ_i}) \in \Omega^{\dim Z_i - 1}(Z_i, o(TZ_i))/d\Omega^{\dim Z_i - 2}(Z_i, o(TZ_i))$  and

$\tilde{\theta}_i(\nabla^F, g^F, g'^F) \in C^\infty(Z_i)$  be the associated Chern-Simons classes of forms such that

$$(0.7) \quad \begin{aligned} d\tilde{e}(TZ_i, \nabla^{TZ_i}, \nabla'^{TZ_i}) &= e(Z_i, \nabla^{TZ_i}) - e(Z_i, \nabla'^{TZ_i}), \\ d\tilde{\theta}_i(\nabla^F, g^F, g'^F) &= \theta_i(\nabla^F, g^F) - \theta_i(\nabla^F, g'^F). \end{aligned}$$

In Section 4, we show:

**Theorem 0.3.** *The following identity holds:*

$$(0.8) \quad \log \left( \frac{\|\cdot\|_{\lambda}^{\text{RS},2}}{\|\cdot\|_{\lambda}^{\text{RS},2}} \right) = \sum_{i=0}^{l_0} \frac{1}{m_i} \left( \int_{Z_i} \tilde{\theta}_i(\nabla^F, g^F, g'^F) e(Z_i, \nabla^{TZ_i}) - \int_{Z_i} \theta_i(\nabla^F, g'^F) \tilde{e}(TZ_i, \nabla^{TZ_i}, \nabla'^{TZ_i}) \right).$$

The arguments in Section 4 are inspired by Bismut-Lott [BL95, Theorem 3.24], who gave a unified proof for the family local index theorem and the anomaly formula [BZ92, Theorem 0.1]. Conceptually, their proof is simpler and more natural than the original proof given by Bismut-Zhang [BZ92, Section IV]. Also, our proof relies on the finite propagation speeds for the solutions of hyperbolic equations on orbifolds, which is originally due to Ma [Ma05].

If  $Z$  is of odd dimension and orientable, then all the  $Z_i$ , for  $0 \leq i \leq l_0$ , is of odd dimension. By Theorem 0.3, the Ray-Singer metric  $\|\cdot\|_{\lambda}^{\text{RS},2}$  does not depend on  $g^{TZ}$  and the flat metric  $g^F$ ; it becomes a topological invariant.

**0.3. A solution of Fried conjecture on locally symmetric orbifolds.** In [F86, P. 537], Fried raised the question of extending his result [F86, Theorem 1] to hyperbolic orbifolds on the equality between the analytic torsion and the zero value of the Ruelle dynamical zeta function associated to a unitarily flat acyclic vector bundle on hyperbolic manifolds. In Section 6, we extend Fried's result to more general compact odd dimensional<sup>1</sup> locally symmetric orbifolds of the reductive type.

Let  $G$  be a connected real reductive group with Cartan involution  $\theta \in \text{Aut}(G)$ . Let  $K \subset G$  be the fixed point of  $\theta$  in  $G$ , so that  $K$  is a maximal compact subgroup of  $G$ . Let  $\mathfrak{g}$  and  $\mathfrak{k}$  be the Lie algebras of  $G$  and  $K$ . Let  $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$  be the Cartan decomposition. Let  $B$  be an  $\text{Ad}(G)$ -invariant and  $\theta$ -invariant non degenerate bilinear form on  $\mathfrak{g}$  such that  $B|_{\mathfrak{p}} > 0$  and  $B|_{\mathfrak{k}} < 0$ . Recall that an element  $\gamma \in G$  is said to be semisimple if and only if  $\gamma$  can be conjugated to  $e^a k^{-1}$  with  $a \in \mathfrak{p}$ ,  $k \in K$ ,  $\text{Ad}(k)a = a$ . And  $\gamma$  is said to be elliptic if and only if  $\gamma$  can be conjugated into  $K$ . Note that if  $\gamma$  is semisimple, its centralizer  $Z(\gamma)$  in  $G$  is still reductive with maximal compact subgroup  $K(\gamma)$ .

Take  $X = G/K$  to be the associated symmetric space. Then,  $B|_{\mathfrak{p}}$  induces a  $G$ -invariant Riemannian metric  $g^{TX}$  on  $X$  such that  $(X, g^{TX})$  is of non positive sectional curvature. Let  $d_X$  be the Riemannian distance on  $X$ .

Let  $\Gamma \subset G$  be a cocompact discrete subgroup of  $G$ . Set  $Z = \Gamma \backslash G/K$ . Then  $Z$  is a compact orbifold with universal covering orbifold  $X$ . To simplify the notation in Introduction, we assume that  $\Gamma$  acts effectively on  $X$ . Then  $\Gamma$  is the orbifold fundamental group of  $Z$ . Clearly,  $\Gamma$  contains only semisimple elements. Let  $\Gamma_e$  be the subset of  $\Gamma$  consisting of elliptic elements, and let  $\Gamma_+ = \Gamma - \Gamma_e$ . Take  $[\Gamma]$  to be the set of conjugacy classes of  $\Gamma$ . Denote by  $[\Gamma_e] \subset [\Gamma]$  the set of elliptic conjugacy classes, and by  $[\Gamma_+] = [\Gamma] - [\Gamma_e]$ .

<sup>1</sup>The even dimensional case is trivial.

The geodesic flow on the unit tangent bundle of  $Z$  is still well-defined. Proceeding as in the proof for the manifold case [DuKoV79, Proposition 5.15], the fixed points of the geodesic flow consists of a disjoint union of smooth connected compact suborbifold

$$(0.9) \quad \coprod_{[\gamma] \in [\Gamma_+]} B_{[\gamma]}.$$

Moreover,  $B_{[\gamma]}$  is diffeomorphic to  $\Gamma \cap Z(\gamma) \backslash Z(\gamma) / K(\gamma)$ . Also, all the elements in  $B_{[\gamma]}$  have the same length  $l_{[\gamma]} > 0$ . Clearly, the geodesic flow induces a locally free  $\mathbb{S}^1$ -action on  $B_{[\gamma]}$ . By an analogy to the multiplicity  $m_i$  of  $Z_i$  in  $Z \coprod \Sigma Z$ , we can define the multiplicity  $m_{[\gamma]}$  of the quotient orbifold  $\mathbb{S}^1 \backslash B_{[\gamma]}$  (c.f. (6.40)). Denote by  $\chi_{\text{orb}}(\mathbb{S}^1 \backslash B_{[\gamma]}) \in \mathbb{Q}$  the orbifold Euler characteristic number [Sa57, Section 3.3] (c.f. also (3.12)) of  $\mathbb{S}^1 \backslash B_{[\gamma]}$ . In Section 6, we show:

**Theorem 0.4.** *If  $\dim Z$  is odd, and if  $F$  is a unitarily flat orbifold vector bundle on  $Z$  with holonomy  $\rho : \Gamma \rightarrow \text{GL}_r(\mathbb{C})$ , then the dynamical zeta function*

$$(0.10) \quad R_\rho(\sigma) = \exp \left( \sum_{[\gamma] \in [\Gamma_+]} \text{Tr}[\rho(\gamma)] \frac{\chi_{\text{orb}}(\mathbb{S}^1 \backslash B_{[\gamma]})}{m_{[\gamma]}} e^{-\sigma l_{[\gamma]}} \right)$$

*is well-defined and holomorphic on  $\text{Re}(\sigma) \gg 1$ , and extends meromorphically to  $\mathbb{C}$ . There exist explicit constant  $C_\rho \in \mathbb{R}$  with  $C_\rho \neq 0$  and  $r_\rho \in \mathbb{Z}$  (c.f. (6.97)) such that as  $\sigma \rightarrow 0$ ,*

$$(0.11) \quad R_\rho(\sigma) = C_\rho T(F)^2 \sigma^{r_\rho} + \mathcal{O}(\sigma^{r_\rho+1}).$$

*Moreover, if  $H^*(Z, F) = 0$ , we have*

$$(0.12) \quad C_\rho = 1, \quad r_\rho = 0,$$

*so that*

$$(0.13) \quad R_\rho(0) = T(F)^2.$$

The proof of Theorem 0.4 is similar to the one given in [Sh16b], except that in the current case, we also need to take account of the contribution of elliptic orbital integrals in the analytic torsion. On the other hand, let us note that a priori elliptic elements do not contribute to the dynamical zeta function. This seemingly contradictory phenomenon has already appeared in the smooth case. In fact, in the current case, the elliptic and non elliptic orbital integrals are related via functional equations of certain Selberg zeta functions.

We refer the readers to the papers of Giulietti-Liverani-Pollicott [GiLiPo13] and Dyatlov-Zworski [DyZ16a, DyZ16b] for other points of view on the dynamical zeta function on negatively curved manifolds.

Let us also mention Fedosova's recent work [Fe15a, Fe15b, Fe16] on the Selberg zeta function and the asymptotic behavior of the analytic torsion of unimodular flat orbifold vector bundles on hyperbolic orbifolds.

**0.4. Organisation of the article.** This article is organized as follows. In Section 1, we introduce some basic notations on the determinant line and characteristic forms. Also we recall some standard terminology on group actions on topological spaces.

In Section 2, we recall the definition of orbifolds, orbifold vector bundles, and the  $\mathcal{G}$ -path theory of Haefliger [H90]. We show Theorem 0.1.



In Section 3, we explain how to extend the usual differential calculus and Chern-Weil theory on manifolds to orbifolds.

Sections 4 and 5 are devoted to a study of the analytic torsion and Ray-Singer metric on orbifolds. In Section 4, following [BL95], we show in a unified way an orbifold version of Gauss-Bonnet-Chern Theorem and Theorem 0.3. Some estimations on heat kernels are postponed to Section 5.

In Section 6, we study the analytic torsion on locally symmetric orbifold using the Selberg trace formula and Bismut's semisimple orbital integral formula. We show Theorem 0.4.

**0.5. Notations.** In the whole paper, we use the superconnection formalism of Quillen [Q85] and [BeGeVe04, Section 1.3]. Here we just briefly recall that if  $A$  is a  $\mathbf{Z}_2$ -graded algebra, if  $a, b \in A$ , the supercommutator  $[a, b]$  is given by

$$(0.14) \quad [a, b] = ab - (-1)^{\deg a \deg b} ba.$$

If  $B$  is another  $\mathbf{Z}_2$ -graded algebra, we denote by  $A \hat{\otimes} B$  the super tensor algebra. If  $E = E^+ \oplus E^-$  is a  $\mathbf{Z}_2$ -graded vector space, the algebra  $\text{End}(E)$  is  $\mathbf{Z}_2$ -graded. If  $\tau = \pm 1$  on  $E^\pm$ , if  $a \in \text{End}(E)$ , the supertrace  $\text{Tr}_s[a]$  is defined by

$$(0.15) \quad \text{Tr}_s[a] = \text{Tr}[\tau a].$$

We make the convention that  $\mathbf{N} = \{0, 1, 2, \dots\}$  and  $\mathbf{N}^* = \{1, 2, \dots\}$ . If  $A$  is a finite set, we denote by  $|A|$  its cardinality.

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## 1. PRELIMINARY

The purpose of this section is to recall some basic definitions and constructions. This section is organized as follows. In subsection 1.1, we introduce the basic conventions on determinant lines.

In subsection 1.2, we recall some standard terminology of group actions on topological spaces, which will be used in the whole paper.

In subsection 1.3, we recall the Chern-Weil construction on characteristic forms and the associated classes of Chern-Simons forms on manifolds.

**1.1. Determinants.** Let  $V$  be a vector space of finite dimension. We denote by  $V^*$  the dual space of  $V$ , and by  $\Lambda V$  the exterior algebra of  $V$ . Set

$$(1.1) \quad \det V = \Lambda^{\dim V} V.$$

Clearly,  $\det V$  is a line. We use the convention that

$$(1.2) \quad \det 0 = \mathbf{C}.$$

If  $\lambda$  is a line, we denote by  $\lambda^{-1} = \lambda^*$  the dual line.

**1.2. Group actions.** Let  $L$  be a topological group acting continuously on a topological space  $S$ . The action of  $L$  is said to be free if for any  $g \in L$  and  $g \neq 1$ , the set of fixed points of  $g$  in  $X$  is empty. The action of  $L$  is said to be effective if the morphism of groups  $L \rightarrow \text{Homeo}(S)$  is injective, where  $\text{Homeo}(S)$  is the group of homeomorphisms of  $S$ . The action of  $L$  is said to be properly discontinuous if for any  $x \in S$  there is a neighborhood  $U$  of  $x$  such that the set

$$(1.3) \quad \{g \in L : gU \cap U \neq \emptyset\}$$

is finite.

If  $L$  acts on the right (reps. left) on the topological space  $S_0$  (resp.  $S_1$ ), denote by  $S_0/L$  (resp.  $L \backslash S_1$ ) the quotient space, and by  $S_0 \times_L S_1$  the quotient of  $S_0 \times S_1$  by the left action defined by

$$(1.4) \quad g(x_0, x_1) = (x_0 g^{-1}, g x_1), \quad \text{for } g \in L \text{ and } (x_0, x_1) \in S_0 \times S_1.$$

If  $S_2$  is another left  $L$ -space, denote by  $S_1 \times_L S_2$  the quotient of  $S_1 \times S_2$  by the evident left action of  $L$ .

**1.3. Characteristic forms on manifolds.** Let  $S$  be a manifold. Denote by  $(\Omega(S), d^S)$  the de Rham complex of  $S$ , and by  $H^*(S)$  the de Rham cohomology.

Let  $E$  be a real vector bundle of rank  $r$  equipped with a Euclidean metric  $g^E$ . Let  $\nabla^E$  be a metric connection, and let  $R^E = (\nabla^E)^2$  be the curvature of  $\nabla^E$ . Then  $R^E$  is a 2-form on  $S$  with values in antisymmetric endomorphisms of  $E$ .

If  $A$  is an antisymmetric matrix, denote by  $\text{Pf}[A]$  the Pfaffian [BZ92, (3.3)] of  $A$ . Then  $\text{Pf}[A]$  is a polynomial function of  $A$ , which is a square root of  $\det[A]$ . Let  $o(E)$  be the orientation line of  $E$ . The Euler form of  $(E, \nabla^E)$  is given by

$$(1.5) \quad e(E, \nabla^E) = \text{Pf} \left[ \frac{R^E}{2\pi} \right] \in \Omega^r(S, o(E)).$$

The cohomology class  $e(E) \in H^*(S)$  of  $e(E, \nabla^E)$  does not depend on the choice of  $(g^E, \nabla^E)$ . More precisely, if  $g'^E$  is another metric on  $E$ , and if  $\nabla'^E$  is another connection on  $E$  which preserves  $g'^E$ , we can define a class of Chern-Simons  $(r-1)$ -form

$$(1.6) \quad \tilde{e}(E, \nabla^E, \nabla'^E) \in \Omega^{r-1}(S, o(E)) / d\Omega^{r-2}(S, o(E))$$

such that

$$(1.7) \quad d^S \tilde{e}(E, \nabla^E, \nabla'^E) = e(E, \nabla'^E) - e(E, \nabla^E).$$

Let us describe the construction of  $\tilde{e}(E, \nabla^E, \nabla'^E)$ . Take a smooth family  $(g_s^E, \nabla_s^E)_{s \in \mathbf{R}}$  of metrics and metric connections such that

$$(1.8) \quad (g_0^E, \nabla_0^E) = (g^E, \nabla^E), \quad (g_1^E, \nabla_1^E) = (g'^E, \nabla'^E).$$

Set

$$(1.9) \quad \pi : \mathbf{R} \times S \rightarrow S.$$



We equip  $\pi^*E$  with a Euclidean metric  $g^{\pi^*E}$  such that

$$(1.10) \quad g^{\pi^*E}|_{\{s\} \times S} = g_s^E,$$

and with a connection

$$(1.11) \quad \nabla^{\pi^*E} = ds \wedge \left( \frac{d}{ds} + \frac{1}{2} g_s^{E,-1} \frac{d}{ds} g_s^E \right) + \nabla_s^E.$$

Clearly,  $\nabla^{\pi^*E}$  preserves  $g^{\pi^*E}$ . Write

$$(1.12) \quad e(\pi^*E, \nabla^{\pi^*E}) = e(E, \nabla_s^E) + ds \wedge \alpha_s \in \Omega^r(\mathbf{R} \times S, \pi^*o(E)).$$

As  $e(\pi^*E, \nabla^{\pi^*E})$  is closed, by (1.12), for  $s \in \mathbf{R}$ , we have

$$(1.13) \quad \frac{\partial}{\partial s} e(E, \nabla_s^E) = d^S \alpha_s.$$

Then,  $\tilde{e}(E, \nabla^E, \nabla'^E) \in \Omega^{r-1}(S, o(E))/d\Omega^{r-2}(S, o(E))$  is defined by the class of

$$(1.14) \quad \int_0^1 \alpha_s ds \in \Omega^{r-1}(S, o(E)).$$

Note that  $\tilde{e}(E, \nabla^E, \nabla'^E)$  does not depend on the choice of smooth family  $(g_s^E, \nabla_s^E)_{s \in \mathbf{R}}$ . Also, (1.7) is a consequence of (1.8) and (1.13). Clearly,  $\tilde{e}(E, \nabla^E, \nabla'^E) = 0$ , if  $\dim S$  is odd.

Let us recall the definition of the  $\hat{A}$ -form of  $(E, \nabla^E)$ . For  $x \in \mathbf{C}$ , set

$$(1.15) \quad \hat{A}(x) = \frac{x/2}{\sinh(x/2)}.$$

Then  $\hat{A}(x)$  is a holomorphic function on  $\mathbf{C}$ . The  $\hat{A}$ -form of  $(E, \nabla^E)$  is given by

$$(1.16) \quad \hat{A}(E, \nabla^E) = \left[ \det \left( \hat{A} \left( -\frac{R^E}{2i\pi} \right) \right) \right]^{1/2} \in \Omega^*(S).$$

Let  $L$  be a compact Lie group. Assume that  $L$  acts fiberwisely and linearly on the vector bundle  $E$  over  $S$ , which preserves  $(g^E, \nabla^E)$ . Take  $g \in L$ . Let  $E(g)$  be the subbundle of  $E$  defined by the fixed points of  $g$ . Let  $\pm\theta_1, \dots, \pm\theta_{s_0}$ ,  $0 < \theta_i \leq \pi$  be the distinct nonzero angles of the action of  $g$  on  $E$ . Let  $E_{\theta_i}$  be the subbundle of  $E$  on which  $g$  acts by a rotation of angle  $\theta_i$ . The subbundle  $E(g)$  and  $E_{\theta_i}$  are canonically equipped with Euclidean metrics and metric connections  $\nabla^{E(g)}, \nabla^{E_{\theta_i}}$ .

For  $\theta \in \mathbf{R} - 2\pi\mathbf{Z}$ , set

$$(1.17) \quad \hat{A}_\theta(x) = \frac{1}{2 \sinh \left( \frac{x+i\theta}{2} \right)}.$$

Given  $\theta_i$ , let  $\hat{A}_{\theta_i}(E_{\theta_i}, \nabla^{E_{\theta_i}})$  be the corresponding multiplicative genus. The equivariant  $\hat{A}$ -form of  $(E, \nabla^E)$  is given by

$$(1.18) \quad \hat{A}_g(E, \nabla^E) = \hat{A}(E(g), \nabla^{E(g)}) \prod_{i=1}^{s_0} \hat{A}_{\theta_i}(E_{\theta_i}, \nabla^{E_{\theta_i}}) \in \Omega^*(S).$$

Let  $E'$  be a complex vector bundle carrying a connection  $\nabla^{E'}$  with curvature  $R^{E'}$ . Assume that  $E'$  is equipped with a fiberwise linear action of  $L$ , which preserves  $\nabla^{E'}$ . For

$g \in L$ , the equivariant Chern character form of  $(E', \nabla^{E'})$  is given by

$$(1.19) \quad \text{ch}_g(E', \nabla^{E'}) = \text{Tr} \left[ g \exp \left( -\frac{R^{E'}}{2i\pi} \right) \right] \in \Omega^{\text{even}}(S).$$

As before,  $\widehat{A}_g(E, \nabla^E)$ ,  $\text{ch}_g(E', \nabla^{E'})$  are closed. Their cohomology classes do not depend on the choice of connections. The closed forms in (1.18) and (1.19) on  $S$  are exactly the ones that appear in the Lefschetz fixed point formula of Atiyah-Bott [ABo67, ABo68]. Note that there are questions of signs to be taken care of, because of the need to distinguish between  $\theta_i$  and  $-\theta_i$ . We refer to the above references for more detail.

Let  $F$  be a flat vector bundle on  $S$  with flat connection  $\nabla^F$ . Let  $g^F$  be a Hermitian metric on  $F$ . Assume that  $F$  is equipped with a fiberwise and linear action of  $L$  which preserves  $\nabla^F$  and  $g^F$ . Put

$$(1.20) \quad \omega(\nabla^F, g^F) = g^{F,-1} \nabla^F g^F.$$

Then,  $\omega(\nabla^F, g^F)$  is a 1-form on  $S$  with values in symmetric endomorphisms of  $F$ . For  $x \in \mathbf{C}$ , set

$$(1.21) \quad h(x) = x e^{x^2}.$$

For  $g \in L$ , the odd Chern character form of  $(F, \nabla^F)$  is given by

$$(1.22) \quad h_g(\nabla^F, g^F) = \sqrt{2i\pi} \text{Tr} \left[ gh \left( \frac{\omega(\nabla^F, g^F)}{\sqrt{2i\pi}} \right) \right] \in \Omega^{\text{odd}}(S).$$

When  $g = 1$ , we denote by  $h(\nabla^F, g^F) = h_1(\nabla^F, g^F)$ .

As before, the cohomology class  $h_g(\nabla^F) \in H^{\text{odd}}(S)$  of  $h_g(\nabla^F, g^F)$  does not depend on  $g^F$ . If  $g'^F$  is another  $L$ -invariant Hermitian metric on  $F$ , we can define the class of Chern-Simons form  $\widetilde{h}_g(\nabla^F, g^F, g'^F) \in \Omega^{\text{even}}(S)/d\Omega^{\text{odd}}(S)$  such that

$$(1.23) \quad d^S \widetilde{h}_g(\nabla^F, g^F, g'^F) = h_g(\nabla^F, g'^F) - h_g(\nabla^F, g^F).$$

More precisely, choose a smooth family of  $L$ -invariant metrics  $(g_s^F)_{s \in \mathbf{R}}$  such that

$$(1.24) \quad g_0^F = g^F, \quad g_1^F = g'^F.$$

Consider the projection  $\pi$  defined in (1.9), and equip  $\pi^*F$  with the pull back flat connection

$$(1.25) \quad \nabla^{\pi^*F} = d^{\mathbf{R}} + \nabla^F,$$

and with the Hermitian metric  $g^{\pi^*F}$  defined by

$$(1.26) \quad g^{\pi^*F} \Big|_{\{s\} \times S} = g_s^F.$$

Write

$$(1.27) \quad h_g(\nabla^{\pi^*F}, g^{\pi^*F}) = h_g(\nabla^F, g_s^F) + ds \wedge \beta_s \in \Omega^{\text{odd}}(\mathbf{R} \times S).$$

As (1.14),  $\widetilde{h}_g(\nabla^F, g^F, g'^F) \in \Omega^{\text{even}}(S)/d\Omega^{\text{odd}}(S)$  is defined by the class of

$$(1.28) \quad \int_0^1 \beta_s ds \in \Omega^{\text{even}}(S).$$

As before,  $\widetilde{h}_g(\nabla^F, g^F, g'^F)$  does not depend on the choice of the smooth family of metrics  $(g_s^F)_{s \in \mathbf{R}}$ . Also,  $\widetilde{h}_g(\nabla^F, g^F, g'^F)$  satisfies (1.23).

## 2. TOPOLOGY OF ORBIFOLDS

The purpose of this section is to introduce some basic definitions and related constructions for orbifolds. We show Theorem 0.1, which claims a bijection between the isomorphism classes of proper flat orbifold vector bundles and the equivalent classes of representations of the orbifold fundamental group.

This section is organized as follows. In subsection 2.1, we recall the definition of orbifolds and the associated groupoid  $\mathcal{G}$ .

In subsection 2.2, we introduce the resolution for the singular part of an orbifold.

In subsection 2.3, we recall the definition of orbifold vector bundles.

In subsection 2.4, the orbifold fundamental group and the universal covering orbifold are constructed using the  $\mathcal{G}$ -path theory of Haefliger [H90, BrH99].

Finally, in subsection 2.5, we define the holonomy representation for a proper flat orbifold vector bundle. We restate and show Theorem 0.1.

**2.1. Definition of orbifolds.** In this subsection, we recall the definition of orbifolds following [Sa57, Section 1] and [AdLeRu07, Section 1.1].

Let  $Z$  be a topological space, and let  $U \subset Z$  be a connected open subset of  $Z$ . Take  $m \in \mathbb{N}$ .

**Definition 2.1.** An  $m$ -dimensional orbifold chart for  $U$  is a triple  $(\tilde{U}, G_U, \pi_U)$ , where

- $\tilde{U} \subset \mathbb{R}^m$  is a connected open subset of  $\mathbb{R}^m$ ;
- $G_U$  is a finite group acting smoothly and effectively on the left on  $\tilde{U}$ ;
- $\pi_U : \tilde{U} \rightarrow U$  is a  $G_U$ -invariant continuous map which induces a homeomorphism of topological spaces

$$(2.1) \quad G_U \backslash \tilde{U} \simeq U.$$

*Remark 2.2.* In [Sa57, Section 1], it is assumed that the codimension of the fixed point set of  $G_U$  in  $\tilde{U}$  is bigger than or equal to 2. In this article, we do not make this assumption.

Let  $U \hookrightarrow V$  be an embedding of connected open subsets of  $Z$ , and let  $(\tilde{U}, G_U, \pi_U)$  and  $(\tilde{V}, G_V, \pi_V)$  be respectively orbifold charts for  $U$  and  $V$ .

**Definition 2.3.** An embedding of orbifold charts is a smooth embedding  $\phi_{VU} : \tilde{U} \rightarrow \tilde{V}$  such that the diagram

$$(2.2) \quad \begin{array}{ccc} \tilde{U} & \xrightarrow{\phi_{VU}} & \tilde{V} \\ \downarrow \pi_U & & \downarrow \pi_V \\ U & \hookrightarrow & V \end{array}$$

commutes.

We recall the following proposition. The proof was given by Satake [Sa57, Lemmas 1.1, 1.2] under the assumption that the codimension of the fixed point set is bigger than or equal to 2. For general cases, see [MoePr97, Appendix] for example.

**Proposition 2.4.** Let  $\phi_{VU} : (\tilde{U}, G_U, \pi_U) \hookrightarrow (\tilde{V}, G_V, \pi_V)$  be an embedding of orbifold charts. The following statements hold:

- (1) if  $g \in G_V$ , then  $x \in \tilde{U} \rightarrow g\phi_{VU}(x) \in \tilde{V}$  is another embedding of orbifold charts. Conversely, any embedding of orbifold charts  $(\tilde{U}, G_U, \pi_U) \hookrightarrow (\tilde{V}, G_V, \pi_V)$  is of such form;
- (2) there exists a unique injective morphism  $\lambda_{VU} : G_U \rightarrow G_V$  of groups such that  $\phi_{VU}$  is  $\lambda_{VU}$ -equivariant;
- (3) if  $g \in G_V$  such that  $\phi_{VU}(\tilde{U}) \cap g\phi_{VU}(\tilde{U}) \neq \emptyset$ , then  $g$  is in the image of  $\lambda_{VU}$ , and so  $\phi_{VU}(\tilde{U}) = g\phi_{VU}(\tilde{U})$ .

Let  $U_1, U_2 \subset Z$  be two connected open subsets of  $Z$  with orbifold charts  $(\tilde{U}_1, G_{U_1}, \pi_{U_1})$  and  $(\tilde{U}_2, G_{U_2}, \pi_{U_2})$ .

**Definition 2.5.** The orbifold charts  $(\tilde{U}_1, G_{U_1}, \pi_{U_1})$  and  $(\tilde{U}_2, G_{U_2}, \pi_{U_2})$  are said to be compatible if for any  $z \in U_1 \cap U_2$ , there is an open connected neighborhood  $U_0 \subset U_1 \cap U_2$  of  $z$  with orbifold chart  $(\tilde{U}_0, G_{U_0}, \pi_{U_0})$  such that there exist two embeddings of orbifold charts

$$(2.3) \quad \phi_{U_i U_0} : (\tilde{U}_0, G_{U_0}, \pi_{U_0}) \hookrightarrow (\tilde{U}_i, G_{U_i}, \pi_{U_i}), \quad \text{for } i = 1, 2.$$

The diffeomorphism  $\phi_{U_2 U_0} \phi_{U_1 U_0}^{-1} : \phi_{U_1 U_0}(\tilde{U}_0) \rightarrow \phi_{U_2 U_0}(\tilde{U}_0)$  is called a coordinate transformation.

**Definition 2.6.** An orbifold atlas on  $Z$  consists of an open connected cover  $\mathcal{U} = \{U\}$  of  $Z$  and a family of compatible orbifold charts  $\tilde{\mathcal{U}} = \{(\tilde{U}, G_U, \pi_U)\}_{U \in \mathcal{U}}$ .

An orbifold atlas  $(\mathcal{V}, \tilde{\mathcal{V}})$  is called a refinement of  $(\mathcal{U}, \tilde{\mathcal{U}})$ , if  $\mathcal{V}$  is a refinement of  $\mathcal{U}$  and if every orbifold chart in  $\tilde{\mathcal{V}}$  has an embedding into some orbifold chart in  $\tilde{\mathcal{U}}$ .

Two orbifold atlases are said to be equivalent if they have a common refinement.

The equivalent class of an orbifold atlas is called an orbifold structure on  $Z$ .

**Definition 2.7.** An orbifold is a second countable Hausdorff space equipped with an orbifold structure. It said to be dimension  $m$ , if all the orbifold charts which define the orbifold structure are of dimension  $m$ .

*Remark 2.8.* Let  $U, V$  be two connected open subsets of an orbifold with respectively orbifold charts  $(\tilde{U}, G_U, \pi_U)$  and  $(\tilde{V}, G_V, \pi_V)$ , which are compatible with the orbifold structure. If  $U \subset V$ , and if  $\tilde{U}$  is simply connected, then there exists an embedding of orbifold charts  $(\tilde{U}, G_U, \pi_U) \hookrightarrow (\tilde{V}, G_V, \pi_V)$ .

*Remark 2.9.* For any point  $z$  of an orbifold, there exists an open connected neighborhood  $U_z \subset Z$  of  $z$  with a compatible orbifold chart  $(\tilde{U}_z, G_z, \pi_z)$  such that  $\pi_z^{-1}(z)$  contains only one point  $x$ . Such chart is called centered at  $x$ . Clearly,  $x$  is a fixed point of  $G_z$ . The isomorphism class of the group  $G_z$  does not depend on the different choices of orbifold charts, and is called the local group at  $z$ .

Moreover, we can choose  $(\tilde{U}_z, G_z, \pi_z)$  to be a linear chart centered at 0, which means

$$(2.4) \quad \tilde{U}_z = \mathbf{R}^m, \quad x = 0 \in \mathbf{R}^m, \quad G_z \subset O(m).$$

In the sequel, let  $Z$  be an orbifold with orbifold atlas  $(\mathcal{U}, \tilde{\mathcal{U}})$ . We assume that  $\mathcal{U}$  is countable and that all the  $\tilde{U} \in \tilde{\mathcal{U}}$  are simply connected. When we talk of an orbifold chart, we mean the one which is compatible with  $\tilde{\mathcal{U}}$ .

Let us introduce a groupoid  $\mathcal{G}$  associated to the orbifold  $Z$  with orbifold atlas  $(\mathcal{U}, \tilde{\mathcal{U}})$ . Recall that a groupoid is a category whose morphisms, which are called arrows, are

isomorphisms. We define  $\mathcal{G}_0$ , the objects of  $\mathcal{G}$ , to be the countable disjoint union of smooth manifold

$$(2.5) \quad \mathcal{G}_0 = \coprod_{U \in \mathcal{U}} \tilde{U}.$$

An arrow  $g$  from  $x_1 \in \mathcal{G}_0$  to  $x_2 \in \mathcal{G}_0$ , denoted by  $g : x_1 \rightarrow x_2$ , is a germ of coordinate transformation  $g$  defined near  $x_1$  such that  $g(x_1) = x_2$ . We denote by  $\mathcal{G}_1$  the set of all arrows. This way defines a groupoid  $\mathcal{G} = (\mathcal{G}_0, \mathcal{G}_1)$ . By [AdLeRu07, Section 1.4],  $\mathcal{G}_1$  is equipped with a topology such that  $\mathcal{G}$  is a proper, effective, étale Lie groupoid.

For  $x_1, x_2 \in \mathcal{G}_0$ , we call  $x_1$  and  $x_2$  in the same orbit if there is an arrow  $g \in \mathcal{G}_1$  from  $x_1$  to  $x_2$ . We denote by  $\mathcal{G}_0/\mathcal{G}_1$  the orbit space equipped with the quotient topology. The projection  $\pi_U : \tilde{U} \rightarrow U$  induces a homeomorphism of topological spaces

$$(2.6) \quad \mathcal{G}_0/\mathcal{G}_1 \simeq Z.$$

Let  $Y$  and  $Z$  be two orbifolds.

**Definition 2.10.** A continuous maps  $f : Y \rightarrow Z$  between orbifolds is called smooth if for any  $y \in Y$ , there exist

- an open connected neighborhood  $U \subset Y$  of  $y$ , an open connected neighborhood  $V \subset Z$  of  $f(y)$  such that  $f(U) \subset V$ ,
- orbifold charts  $(\tilde{U}, G_U, \pi_U)$  and  $(\tilde{V}, G_V, \pi_V)$  for  $U$  and  $V$ ,
- a smooth map  $\tilde{f}_U : \tilde{U} \rightarrow \tilde{V}$  such that the following diagram

$$(2.7) \quad \begin{array}{ccc} \tilde{U} & \xrightarrow{\tilde{f}_U} & \tilde{V} \\ \downarrow \pi_U & & \downarrow \pi_V \\ U & \xrightarrow{f|_U} & V \end{array}$$

commutes.

We denote by  $C^\infty(Y, Z)$  the space of smooth maps from  $Y$  to  $Z$ .

Two orbifolds  $Y$  and  $Z$  are called isomorphic if there are smooth maps  $f : Y \rightarrow Z$  and  $f' : Z \rightarrow Y$  such that  $ff' = 1$  and  $f'f = 1$ . Clearly, this is the case if  $f : Y \rightarrow Z$  is a smooth homeomorphism such that each lifting  $\tilde{f}_U$  is a diffeomorphism. Moreover, in this case, by Proposition 2.4, there is an isomorphism of group  $\rho_U : G_U \rightarrow G_V$  such that  $\tilde{f}_U$  is  $\rho_U$ -equivariant. Also, any possible lifting has the form  $g\tilde{f}_U$ ,  $g \in G_U$ .

**Definition 2.11.** An action of Lie group  $L$  on  $Z$  is said to be smooth, if the action  $L \times Z \rightarrow Z$  is smooth.

The following proposition is an extension of [T80, Proposition 13.2.1]. We include a detailed proof as some constructions in the proof will be useful to show Theorem 0.1.

**Proposition 2.12.** *Let  $\Gamma$  be a discrete group acting smoothly and properly discontinuously on the left on an orbifold  $X$ . Then  $\Gamma \backslash X$  has a canonical orbifold structure induced from  $X$ .*

*Proof.* Let  $p : X \rightarrow \Gamma \backslash X$  be the natural projection. We equip  $\Gamma \backslash X$  with the quotient topology. Since  $X$  is Hausdorff and second countable, and since the  $\Gamma$ -action is properly discontinuous, then  $\Gamma \backslash X$  is also Hausdorff and second countable.

Take  $z \in \Gamma \backslash X$ . We choose  $x \in X$  such that  $p(x) = z$ . Set

$$(2.8) \quad \Gamma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

As the  $\Gamma$ -action is properly discontinuous,  $\Gamma_x$  is a finite group, and there exists an open connected  $\Gamma_x$ -invariant neighborhood  $V_x \subset X$  of  $x$  such that for  $\gamma \in \Gamma - \Gamma_x$ ,

$$(2.9) \quad \gamma V_x \cap V_x = \emptyset.$$

Then,  $p(V_x) \subset \Gamma \backslash X$  is an open connected neighborhood of  $z$ . Also, we have

$$(2.10) \quad \Gamma_x \backslash V_x \simeq \Gamma \backslash \Gamma V_x = p(V_x).$$

By taking  $V_x$  small enough, there is an orbifold chart  $(\tilde{V}_x, H_x, \pi_x)$  for  $V_x$  centered at  $\tilde{x} \in \tilde{V}_x$  (c.f. Remark 2.9). As  $\Gamma$  acts smoothly on  $X$ , we can assume that homeomorphism of  $V_x$  defined by  $\gamma_x \in \Gamma_x$  lifts to a local diffeomorphism  $\tilde{\gamma}_x$  defined near  $\tilde{x}$  such that  $\pi_x \tilde{\gamma}_x = \gamma_x \pi_x$  holds near  $\tilde{x}$ .

By Proposition 2.4, the lifting  $\tilde{\gamma}_x$  is not unique, and all possible liftings can be written as  $h_x \tilde{\gamma}_x$  for some  $h_x \in H_x$ . Let  $G_x$  be the group of local diffeomorphism defined near  $x$  generated by  $\{\tilde{\gamma}_x\}_{\gamma_x \in \Gamma_x}$  and  $H_x$ . Then  $G_x$  is a finite group. By choosing  $V_x$  small enough and by (2.10),  $G_x$  acts on  $\tilde{V}_x$  such that

$$(2.11) \quad G_x \backslash \tilde{V}_x \simeq p(V_x).$$

As the  $G_x$ -action on  $\tilde{V}_x$  is effective,  $(\tilde{V}_x, G_x, p \circ \pi_x)$  is an orbifold chart of  $Z$  for  $p(V_x)$ .

The family of open sets  $\{p(V_x)\}$  covers  $\Gamma \backslash X$ . It remains to show that two such orbifold charts  $(\tilde{V}_{x_1}, G_{x_1}, p \circ \pi_{x_1})$  and  $(\tilde{V}_{x_2}, G_{x_2}, p \circ \pi_{x_2})$  are compatible. Its proof consists of two steps.

In the first step, we consider the case  $x_2 = \gamma x_1$  for some  $\gamma \in \Gamma$ . We can assume that  $V_{x_2} = \gamma V_{x_1}$ , and that  $\gamma|_{V_{x_1}}$  lifts to  $\tilde{\gamma}_{x_1} : \tilde{V}_{x_1} \rightarrow \tilde{V}_{x_2}$ . Then  $\tilde{\gamma}_{x_1}$  defines an isomorphism between the orbifold charts  $(\tilde{V}_{x_1}, G_{x_1}, p \circ \pi_{x_1})$  and  $(\tilde{V}_{x_2}, G_{x_2}, p \circ \pi_{x_2})$ .

In the second step, we consider general  $x_1, x_2 \in X$  such that  $p(V_{x_1}) \cap p(V_{x_2}) \neq \emptyset$ . Because of the first step, we can assume that  $V_{x_1} \cap V_{x_2} \neq \emptyset$ . For  $x_0 \in V_{x_1} \cap V_{x_2}$ , take an open connected neighborhood  $V_{x_0} \subset V_{x_1} \cap V_{x_2}$  of  $x_0$  and an orbifold chart  $(\tilde{V}_{x_0}, H_{x_0}, \pi_{x_0})$  of  $X$  as before. We can assume that there exist two embeddings  $\phi_{V_{x_i} V_{x_0}} : (V_{x_0}, H_{x_0}, \pi_{x_0}) \hookrightarrow (\tilde{V}_{x_i}, H_{x_i}, \pi_{x_i})$ , for  $i = 1, 2$ , of orbifold charts of  $X$ . Then,  $\phi_{V_{x_i} V_{x_0}}$  also define two embeddings of orbifold charts of  $Z$ ,

$$(2.12) \quad (\tilde{V}_{x_0}, G_{x_0}, p \circ \pi_{x_0}) \hookrightarrow (\tilde{V}_{x_i}, G_{x_i}, p \circ \pi_{x_i}).$$

The proof of Proposition 2.12 is completed.  $\square$

*Remark 2.13.* By the construction,  $H_x$  is a normal subgroup of  $G_x$ , and  $\gamma_x \rightarrow \tilde{\gamma}_x$  induces a surjective morphism of groups

$$(2.13) \quad \Gamma_x \rightarrow G_x / H_x.$$

If the action of  $\Gamma$  on  $X$  is effective, then (2.13) is an isomorphism of group such that we have the exact sequence of groups

$$(2.14) \quad 1 \rightarrow H_x \rightarrow G_x \rightarrow \Gamma_x \rightarrow 1.$$



**2.2. Singular points of orbifolds.** Let  $Z$  be an orbifold with orbifold atlas  $(\mathcal{U}, \tilde{\mathcal{U}})$ . Put

$$(2.15) \quad Z_{\text{reg}} = \{z \in Z : G_z = \{1\}\}, \quad Z_{\text{sing}} = \{z \in Z : G_z \neq \{1\}\}.$$

Then  $Z = Z_{\text{reg}} \cup Z_{\text{sing}}$ . Clearly,  $Z_{\text{reg}}$  is a smooth manifold. However,  $Z_{\text{sing}}$  is not necessarily an orbifold. Following [K78, Section 1], we will introduce the orbifold resolution  $\Sigma Z$  for  $Z_{\text{sing}}$ .

Let  $[G_z]$  be the set of conjugacy classes of  $G_z$ . Set

$$(2.16) \quad \Sigma Z = \{(z, [g]) : z \in Z, [g] \in [G_z] \setminus \{1\}\}.$$

By [K78, Section 1],  $\Sigma Z$  possess a natural orbifold structure. Indeed, take  $U \in \mathcal{U}$  and  $(\tilde{U}, \pi_U, G_U) \in \tilde{\mathcal{U}}$ . For  $g \in G_U$ , denote by  $\tilde{U}^g \subset \tilde{U}$  the set of fixed points of  $g$  in  $\tilde{U}$ , and by  $Z_{G_U}(g) \subset G_U$  the centralizer of  $g$  in  $G_U$ . Clearly,  $Z_{G_U}(g)$  acts on  $\tilde{U}^g$ , and the quotient  $Z_{G_U}(g) \backslash \tilde{U}^g$  depends only on the conjugacy class of  $g$  in  $G_U$ . The map  $x \in \tilde{U}^g \rightarrow (\pi_U(x), [g]) \in \Sigma U$  induces an identification

$$(2.17) \quad \coprod_{[g] \in [G_U] \setminus \{1\}} Z_{G_U}(g) \backslash \tilde{U}^g \simeq \Sigma U.$$

We equip  $\Sigma U$  with the topology and the orbifold structure via (2.17). The topology and the orbifold structure on  $\Sigma Z$  is obtained by gluing  $\Sigma U$ . We omit the detail.

We decompose  $\Sigma Z = \coprod_{i=1}^{l_0} Z_i$  following its connected components. If  $(z, [g]) \in Z_i$ , set

$$(2.18) \quad m_i = \left| \ker (Z_{G_U}(g) \rightarrow \text{Diffeo}(\tilde{U}^g)) \right| \in \mathbb{N}.$$

By definition,  $m_i$  is locally constant, and is called the multiplicity of  $Z_i$ . In the sequel, we will use the notation

$$(2.19) \quad Z_0 = Z, \quad m_0 = 1.$$

**2.3. Orbifold vector bundle.** We recall the definition of orbifold vector bundles.

**Definition 2.14.** A complex orbifold vector bundle  $E$  of rank  $r$  on  $Z$  consists of an orbifold  $\mathcal{E}$ , called the total space, and a smooth map  $\pi : \mathcal{E} \rightarrow Z$ , such that

- (1) there is an orbifold atlas  $(\mathcal{U}, \tilde{\mathcal{U}})$  of  $Z$  such that for any  $U \in \mathcal{U}$  and  $(\tilde{U}, G_U, \pi_U) \in \tilde{\mathcal{U}}$ , there exist a finite group  $G_U^E$  acting smoothly on  $\tilde{U}$  which induces a surjective morphism of groups  $G_U^E \rightarrow G_U$ , a representation  $\rho_U^E : G_U^E \rightarrow \text{GL}_r(\mathbb{C})$  of  $G_U^E$ , and a  $G_U^E$ -invariant continuous map  $\pi_U^E : \tilde{U} \times \mathbb{C}^r \rightarrow \pi^{-1}(U)$  which induces an homomorphism of topological spaces

$$(2.20) \quad \tilde{U} \times_{G_U^E} \mathbb{C}^r \simeq \pi^{-1}(U);$$

- (2) the triple  $(\tilde{U} \times \mathbb{C}^r, G_U^E, \pi_U^E)$  is a (compatible) orbifold chart on  $\mathcal{E}$ ;
- (3) if  $U_1, U_2 \in \mathcal{U}$  such that  $U_1 \cap U_2 \neq \emptyset$ , and for any  $z \in U_1 \cap U_2$ , there exist a connected open neighborhood  $U_0 \subset U_1 \cap U_2$  of  $z$  with a simply connected orbifold chart  $(\tilde{U}_0, G_{U_0}, \pi_{U_0})$ ,  $(G_{U_0}^E, \rho_{U_0}^E : G_{U_0}^E \rightarrow \text{GL}_r(\mathbb{C}), \pi_{U_0}^E : \tilde{U}_0 \times \mathbb{C}^r \rightarrow \pi^{-1}(U_0))$  such that (1) and (2) hold, and the embeddings of orbifold charts

$$(2.21) \quad \phi_{U_i U_0}^E : (\tilde{U}_0 \times \mathbb{C}^r, G_{U_0}^E, \pi_{U_0}^E) \hookrightarrow (\tilde{U}_i \times \mathbb{C}^r, G_{U_i}^E, \pi_{U_i}^E), \quad \text{for } i = 1, 2,$$

which have the following form

$$(2.22) \quad (x, v) \in \tilde{U}_0 \times \mathbb{C}^r \rightarrow (\phi_{U_i U_0}(x), g_{U_i U_0}^E(x)v) \in \tilde{U}_i \times \mathbb{C}^r, \quad \text{for } i = 1, 2,$$

where  $\phi_{U_i U_0} : (\tilde{U}_0, G_{U_0}, \pi_{U_0}) \hookrightarrow (\tilde{U}_i, G_{U_i}, \pi_{U_i})$  is an embedding of orbifold charts of  $Z$ , and  $g_{U_i U_0}^E \in C^\infty(\tilde{U}_0, \mathrm{GL}_r(\mathbf{C}))$ .

The vector bundle  $E$  is called proper if the surjective morphism  $G_U^E \rightarrow G_U$  is an isomorphism, and is called flat if all the  $g_{U_i U_0}^E$  can be chosen to be constant functions.

*Remark 2.15.* The embedding  $\phi_{U_i U_0}^E$  exists as  $\tilde{U}_0 \times \mathbf{C}^r$  is simply connected. By Proposition 2.4, it is uniquely determined by the first component  $\phi_{U_i U_0}$  when  $E$  is proper.

*Remark 2.16.* We can define the real orbifold vector bundle in an obvious way.

In the sequel, for  $U \in \mathcal{U}$ , we will denote by  $\tilde{E}_U$  the trivial vector bundle of rank  $r$  on  $\tilde{U}$ , and by  $E_U$  the restriction of  $E$  to  $U$ . Their total spaces are given respectively by

$$(2.23) \quad \tilde{\mathcal{E}}_U = \tilde{U} \times \mathbf{C}^r, \quad \mathcal{E}_U = \tilde{U} \times_{G_U} \mathbf{C}^r.$$

Let us identify the associated groupoid  $\mathcal{G}^E = (\mathcal{G}_0^E, \mathcal{G}_1^E)$  for the total space  $\mathcal{E}$  of a proper orbifold vector bundle  $E$ . By (2.5), the object of  $\mathcal{G}^E$  is given by

$$(2.24) \quad \mathcal{G}_0^E = \coprod_{U \in \mathcal{U}} \tilde{\mathcal{E}}_U = \mathcal{G}_0 \times \mathbf{C}^r.$$

If  $g \in \mathcal{G}_1$  is represented by the germ of the function  $\phi_{U_2 U_0} \phi_{U_1 U_0}^{-1}$ , denote by  $g_*^E$  the germ of function  $g_{U_2 U_0}^E g_{U_1 U_0}^{E, -1}$ . By Remark 2.15,  $g_*^E$  is uniquely determined by  $g$ . Thus, if  $g$  is an arrow from  $x$ , and if  $v \in \mathbf{C}^r$ ,  $(g, v)$  defines an arrow from  $(x, v)$  to  $(gx, g_*^E v)$ . This way gives the identification

$$(2.25) \quad \mathcal{G}_1^E = \mathcal{G}_1 \times \mathbf{C}^r.$$

We give some examples of orbifold vector bundles.

**Example 2.17.** The tangent bundle  $TZ$  of an orbifold  $Z$  is a real proper orbifold vector bundle locally defined by  $\{(T\tilde{U}, G_U)\}_{U \in \mathcal{U}}$ .

**Example 2.18.** Assume  $Z$  is covered by linear charts  $\{(\tilde{U}, G_U, \pi_U)\}_{U \in \mathcal{U}}$  (c.f. Remark 2.9). The orientation line  $o(TZ)$  is a real proper orbifold line bundle on  $Z$ , locally defined by  $(\tilde{U} \times \mathbf{R}, G_U)$  where the action of  $g \in G_U$  is given by

$$(2.26) \quad g : (x, v) \in \tilde{U} \times \mathbf{R} \rightarrow (gx, \det(g)v) \in \tilde{U} \times \mathbf{R}.$$

Clearly,  $o(TZ)$  is flat. If  $o(TZ)$  is trivial,  $Z$  is called orientable.

**Example 2.19.** If  $E, F$  are orbifold vector bundles on  $Z$ , then  $E^*$ ,  $\Lambda^*(E)$  and  $E \otimes F$  are defined in an obvious way.

**Example 2.20.** Let  $E$  be an orbifold vector bundle on  $Z$ . For  $U \in \mathcal{U}$ , let  $V_U \subset \mathbf{C}^r$  be the maximum  $\ker(G_U^E \rightarrow G_U)$ -invariant subspace of  $\mathbf{C}^r$ . Then  $G_U$  acts on  $V_U$ , and  $\{(\tilde{U} \times V_U, G_U)\}_{U \in \mathcal{U}}$  defines a proper orbifold vector bundle  $E^{\mathrm{pr}}$  on  $Z$ . Clearly, if  $E$  is flat, then  $E^{\mathrm{pr}}$  is also flat.

**Definition 2.21.** The space of smooth of sections of  $E$  is defined by

$$(2.27) \quad C^\infty(Z, E) = \{s \in C^\infty(Z, \mathcal{E}) : \pi \circ s = \mathrm{id}\}.$$

By definition, we have

$$(2.28) \quad C^\infty(Z, E) = C^\infty(Z, E^{\text{pr}}).$$

For this reason, in the rest of this section, unless otherwise specified, all the vector bundle are assumed to be proper.

By (2.6),  $s \in C^\infty(Z, E)$  can be represented by a family  $\{s_U \in C^\infty(\tilde{U}, \tilde{E}_U)^{G_U}\}_{U \in \mathcal{U}}$  of  $G_U$ -invariant sections such that for any  $x_1 \in U_1, x_2 \in U_2$  and  $g \in \mathcal{G}_1$  from  $x_1$  to  $x_2$ , near  $x_1$  we have

$$(2.29) \quad g^* s_{U_1} = s_{U_2}.$$

Similarly, for  $k \in \mathbb{N}$ , the space of  $C^k(Z, E)$  is formed by the continuous maps  $s : Z \rightarrow \mathcal{E}$  such that  $\pi \circ s = \text{id}$ , and the restriction of  $s$  to  $U$  has a  $C^k$ -lifting. In other words, there is a family  $\{s_U \in C^k(\tilde{U}, \tilde{E}_U)^{G_U}\}_{U \in \mathcal{U}}$  of  $G_U$ -invariant sections, such that (2.29) holds.

Two orbifold vector bundles  $E$  and  $F$  are called isomorphic if there is  $f \in C^\infty(Z, E^* \otimes F)$  and  $g \in C^\infty(Z, F^* \otimes E)$  such that  $fg = 1$  and  $gf = 1$ .

Let  $\Gamma$  be a discrete group acting smoothly and properly discontinuously on an orbifold  $X$ . Let  $\rho : \Gamma \rightarrow \text{GL}_r(\mathbb{C})$  be a representation of  $\Gamma$ . By Proposition 2.12,  $\Gamma \backslash X$  and

$$(2.30) \quad \mathcal{F} = X \times_{\Gamma} \mathbb{C}^r$$

have canonical orbifold structures. The first projection  $X \times \mathbb{C}^r \rightarrow X$  descends to a smooth map of orbifold

$$(2.31) \quad \pi : \mathcal{F} \rightarrow \Gamma \backslash X.$$

**Proposition 2.22.** *Assume that the action of  $\Gamma$  on  $X$  is smooth, properly discontinuous and effective. Then (2.31) defines canonically a proper flat vector bundle  $F$  on  $\Gamma \backslash X$ .*

*Proof.* Recall that  $p : X \rightarrow \Gamma \backslash X$  is the projection. For  $x \in X$ , we use the same notations  $\Gamma_x, V_x, (\tilde{V}_x, H_x)$  and  $G_x$  as in the proof of Proposition 2.12. Then,  $\Gamma \backslash X$  is covered by

$$(2.32) \quad p(V_x) \simeq \Gamma_x \backslash V_x \simeq G_x \backslash \tilde{V}_x.$$

The stabilizer subgroup of  $\Gamma$  at  $(x, 0) \in X \times \mathbb{C}^r$  is  $\Gamma_x$ . By (2.9), if  $\gamma \in \Gamma - \Gamma_x$ ,

$$(2.33) \quad \gamma(V_x \times \mathbb{C}^r) \cap (V_x \times \mathbb{C}^r) = \emptyset.$$

As (2.10), we have

$$(2.34) \quad \pi^{-1}(p(V_x)) = \Gamma \backslash \Gamma(V_x \times \mathbb{C}^r) \simeq V_x \times_{\Gamma_x} \mathbb{C}^r.$$

As the action of  $\Gamma$  on  $X$  is effective, by Remark 2.13, we have a morphism of groups  $G_x \rightarrow \Gamma_x$ . The group  $G_x$  acts on  $\mathbb{C}^r$  via the composition of  $G_x \rightarrow \Gamma_x$  and  $\rho|_{\Gamma_x} : \Gamma_x \rightarrow \text{GL}_r(\mathbb{C})$ . Thus,  $G_x$  acts on  $\tilde{V}_x \times \mathbb{C}^r$  effectively such that

$$(2.35) \quad V_x \times_{\Gamma_x} \mathbb{C}^r \simeq \tilde{V}_x \times_{G_x} \mathbb{C}^r.$$

By Proposition 2.12,  $(\tilde{V}_x \times \mathbb{C}^r, G_x)$  is an orbifold chart of  $\mathcal{F}$  for  $\pi^{-1}(p(V_x))$ .

Take two  $(\tilde{V}_{x_1} \times \mathbb{C}^r, G_{x_1})$  and  $(\tilde{V}_{x_2} \times \mathbb{C}^r, G_{x_2})$  orbifold charts of  $\mathcal{F}$ . It remains to show the compatibility condition (2.22). We proceed as in the proof of Proposition 2.12. If  $x_2 = \gamma x_1$ , then

$$(2.36) \quad (x, v) \in \tilde{V}_{x_1} \times \mathbb{C}^r \rightarrow (\tilde{\gamma}_{x_1} x, \rho(\gamma)v) \in \tilde{V}_{x_2} \times \mathbb{C}^r$$

defines an isomorphism of orbifold charts on  $\mathcal{F}$ .

For general  $x_1, x_2 \in X$ , we can assume that  $V_{x_1} \cap V_{x_2} \neq \emptyset$ . For  $x_0 \in V_{x_1} \cap V_{x_2}$ , take  $V_{x_0}$ ,  $\tilde{V}_{x_0}$  and  $\phi_{V_{x_i} V_{x_0}}$  as in the proof of Proposition 2.12. Then

$$(2.37) \quad (x, v) \in \tilde{V}_{x_0} \times \mathbf{C}^r \rightarrow (\phi_{V_{x_i} V_{x_0}}(x), v) \in \tilde{V}_{x_i} \times \mathbf{C}^r$$

define two embeddings of orbifold charts of  $\mathcal{F}$ . From (2.36) and (2.37), we deduce that (2.31) defines a flat orbifold vector bundle on  $\Gamma \backslash X$ . The properness is clear from the construction. The proof of Proposition 2.22 is completed.  $\square$

*Remark 2.23.* Take  $A \in \mathrm{GL}_r(\mathbf{C})$ . Let  $\rho_A : \gamma \in \Gamma \rightarrow A\rho(\gamma)A^{-1} \in \mathrm{GL}_r(\mathbf{C})$  be another representation of  $\Gamma$ . Then

$$(2.38) \quad (x, v) \in X \times \mathbf{C}^r \rightarrow (x, Av) \in X \times \mathbf{C}^r$$

descends to an isomorphism between flat orbifold vector bundles  $X_{\rho} \times \mathbf{C}^r$  and  $X_{\rho_A} \times \mathbf{C}^r$ .

**2.4. Orbifold fundamental groups and universal covering orbifold.** In this subsection, following [H90], [BrH99, Section III.3], we recall the constructions of the orbifold fundamental group and the universal covering orbifold. We assume that the orbifold  $Z$  is connected. Let  $\mathcal{G}$  be the groupoid associated with some orbifold atlas  $(\mathcal{U}, \tilde{\mathcal{U}})$ .

**Definition 2.24.** A continuous  $\mathcal{G}$ -path  $c = (b_1, \dots, b_k; g_0, \dots, g_k)$  parameterized by  $[0, 1]$  starting at  $x \in \mathcal{G}_0$  and ending at  $y \in \mathcal{G}_0$  is given by

- (1) a partition  $0 = t_0 < t_1 < \dots < t_k = 1$  of  $[0, 1]$ ;
- (2) continuous paths  $b_i : [t_{i-1}, t_i] \rightarrow \mathcal{G}_0$ , for  $1 \leq i \leq k$ ;
- (3) arrows  $g_i \in \mathcal{G}_1$  such that  $g_0 : x \rightarrow b_1(0)$ ,  $g_i : b_i(t_i) \rightarrow b_{i+1}(t_i)$ , for  $1 \leq i \leq k-1$ , and  $g_k : b_k(1) \rightarrow y$ .

If  $x = y$ , we call  $c$  is a  $\mathcal{G}$ -loop based at  $x$ .

Two  $\mathcal{G}$ -paths

$$(2.39) \quad c = (b_1, \dots, b_k; g_0, \dots, g_k), \quad c' = (b'_1, \dots, b'_{k'}; g'_0, \dots, g'_{k'}),$$

such that  $c$  ending at  $y$  and  $c'$  starting at  $y$  can be composed into a  $\mathcal{G}$ -path (with a suitable reparameterization c.f. [BrH99, Section III.3.4]),

$$(2.40) \quad cc' = (b_1, \dots, b_k, b'_1, \dots, b'_{k'}; g_0, \dots, g'_0 g_k, \dots, g'_{k'}).$$

Also, we can define the inverse of a  $\mathcal{G}$ -path in an obvious way.

**Definition 2.25.** We define an equivalence relation on  $\mathcal{G}$ -paths generated by

- (1) subdivision of the partition and adjunction by identity elements of  $\mathcal{G}_1$  on new partition points.
- (2) for some  $1 \leq i_0 \leq k$ , replacement of the triple  $(b_{i_0}, g_{i_0-1}, g_{i_0})$  by  $(hb_{i_0}, hg_{i_0-1}, g_{i_0}h^{-1})$ , where  $h \in \mathcal{G}_1$  is well-defined near  $b_{i_0}([t_{i_0-1}, t_{i_0}])$ .

**Definition 2.26.** An elementary homotopy between two  $\mathcal{G}$ -paths  $c$  and  $c'$  is a family, parameterized by  $s \in [0, 1]$ , of  $\mathcal{G}$ -paths  $c^s = (b_1^s, \dots, b_k^s; g_0^s, \dots, g_k^s)$ , over the subdivisions  $0 = t_0^s \leq t_1^s \leq \dots \leq t_k^s = 1$ , where  $t_i^s, b_i^s$  and  $g_i^s$  depend continuously on the parameter  $s$ , the elements  $g_0^s$  and  $g_k^s$  are independent of  $s$  and  $c^0 = c, c^1 = c'$ .

**Definition 2.27.** Two  $\mathcal{G}$ -paths are called to be homotopic (with fixed extremities) if one can pass from the first to the second by equivalences of  $\mathcal{G}$ -paths and elementary homotopies. The homotopy class of a  $\mathcal{G}$ -path  $c$  will be denoted by  $[c]$ .

As ordinary paths in topological spaces, the composition and inverse operations of  $\mathcal{G}$ -paths are well-defined for their homotopy classes.

**Definition 2.28.** Take  $x_0 \in \mathcal{G}_0$ . With the operations of composition and inverse of  $\mathcal{G}$ -paths, the homotopy classes of  $\mathcal{G}$ -loops based at  $x_0$  form a group  $\pi_1^{\text{orb}}(Z, x_0)$  called the orbifold fundamental group.

As  $Z$  is connected, any two points of  $\mathcal{G}_0$  can be connected by a  $\mathcal{G}$ -path. Thus, the isomorphic class of the group  $\pi_1^{\text{orb}}(Z, x_0)$  does not depend on the choice of  $x_0$ . Also, it depends only on the orbifold structure of  $Z$ . In the sequel, for simplicity, we denote by  $\Gamma = \pi_1^{\text{orb}}(Z, x_0)$ .

*Remark 2.29.* As a fundamental group of a manifold,  $\Gamma$  is countable.

In the rest of this subsection, we will construct an orbifold  $X$ , called the universal covering orbifold of  $Z$ . See also in [BrH99, Section III.G.3.20]. The terminology of universal covering orbifold will be explained later in Remark 2.31. Let us begin with introducing a groupoid  $\widehat{\mathcal{G}}$ . Assume that  $\mathcal{U} = \{U_z\}$  and  $\widetilde{\mathcal{U}} = \{(\widetilde{U}_z, G_z, \pi_z)\}$  where all the  $\widetilde{U}_z$  are simply connected and are centered at  $x \in \widetilde{U}_z$  as in Remark 2.9. Fix  $x_0 \in \mathcal{G}_0$  as before.

Let  $\widehat{\mathcal{G}}_0$  be the space of homotopy classes of  $\mathcal{G}$ -paths starting at  $x_0$ . The group  $\Gamma$  acts naturally on  $\widehat{\mathcal{G}}_0$  by composition at the starting point  $x_0$ . We denote by

$$(2.41) \quad \widehat{p}: \widehat{\mathcal{G}}_0 \rightarrow \mathcal{G}_0$$

the projection sending  $[c] \in \widehat{\mathcal{G}}_0$  to its ending point. Clearly,  $\widehat{p}$  is  $\Gamma$ -invariant.

Define a topology and manifold structure on  $\widehat{\mathcal{G}}_0$  as follows. For  $x_1, x_2 \in \widetilde{U}_z$ , we denote by  $c_{x_1 x_2} = (b_{x_1 x_2}; 1, 1)$  a  $\mathcal{G}$ -path starting at  $x_1$  and ending at  $x_2$ , where  $b_{x_1 x_2}$  is a path in  $\widetilde{U}_z$  connecting  $x_1$  and  $x_2$ . Note that since  $\widetilde{U}_z$  is simply connected, the homotopy class  $[c_{x_1 x_2}]$  does not depend on the choice of  $b_{x_1 x_2}$ . For each  $U_z \in \mathcal{U}$ , we fix a  $\mathcal{G}$ -path  $c_z$  starting at  $x_0$  and ending at  $x \in \widetilde{U}_z$ . For  $a \in \Gamma$ , set

$$(2.42) \quad \widetilde{V}_{z,a} = \left\{ [c] \in \widehat{p}^{-1}(\widetilde{U}_z) : [cc_{\widehat{p}([c])x}c_z^{-1}] = a \right\}.$$

By (2.41) and (2.42), we have

$$(2.43) \quad \widehat{p}^{-1}(\widetilde{U}_z) = \coprod_{a \in \Gamma} \widetilde{V}_{z,a}, \quad \widehat{\mathcal{G}}_0 = \coprod_{U_z \in \mathcal{U}, a \in \Gamma} \widetilde{V}_{z,a}.$$

Also,

$$(2.44) \quad \widehat{p}: \widetilde{V}_{z,a} \rightarrow \widetilde{U}_z$$

is a bijection. We equip  $\widetilde{V}_{z,a}$  with a topology and a manifold structure via (2.44). Clearly, the choice of  $c_z$  is irrelevant. By (2.43),  $\widehat{\mathcal{G}}_0$  is a countable disjoint union of smooth manifolds such that (2.41) is a Galois covering with deck transformation group  $\Gamma$ .

If  $y \in \mathcal{G}_0$  and if  $g \in \mathcal{G}_1$  is defined near  $y$ , we denote by

$$(2.45) \quad c_{y,g} = (b_y; 1, g)$$

the  $\mathcal{G}$ -path, where  $b_y$  is the constant path at  $y$ . Set

$$(2.46) \quad \widehat{\mathcal{G}}_1 = \{([c], g) \in \widehat{\mathcal{G}}_0 \times \mathcal{G}_1 : g \text{ is defined near } \widehat{p}([c]) \in \mathcal{G}_0\}.$$

Then,  $([c], g) \in \widehat{\mathcal{G}}_1$  represents an arrow from  $[c]$  to  $[c]c_{\widehat{p}([c],g)}$ . This defines a groupoid  $\widehat{\mathcal{G}} = (\widehat{\mathcal{G}}_0, \widehat{\mathcal{G}}_1)$ .

Let

$$(2.47) \quad X = \widehat{\mathcal{G}}_0 / \widehat{\mathcal{G}}_1$$

be the orbit space of  $\widehat{\mathcal{G}}$  equipped with the quotient topology. The action of  $\Gamma$  on  $\widehat{\mathcal{G}}_0$  descends to an effective and continuous action on  $X$ . The projection  $\widehat{p}$  descends to a  $\Gamma$ -invariant continuous map

$$(2.48) \quad p : X \rightarrow Z.$$

**Theorem 2.30.** *Assume  $Z$  is a connected orbifold. The topological space  $X$  has a canonical orbifold structure such that  $\Gamma$  acts smoothly, effectively and properly discontinuously on  $X$ . Moreover, (2.48) induces an isomorphism of orbifolds*

$$(2.49) \quad \Gamma \backslash X \rightarrow Z.$$

*Proof.* Let us construct an orbifold atlas on  $X$ . For  $a \in \Gamma$ , let  $\pi_{z,a}$  be the composition of continuous maps  $\widetilde{V}_{z,a} \hookrightarrow \widehat{\mathcal{G}}_0 \rightarrow X$ . Set

$$(2.50) \quad V_{z,a} = \pi_{z,a}(\widetilde{V}_{z,a}) \subset X.$$

By (2.43),

$$(2.51) \quad p^{-1}(U_z) = \bigcup_{a \in \Gamma} V_{z,a}.$$

Recall  $x \in \widetilde{U}_z$  and  $\pi_z(x) = z$ . By (2.45), for  $g \in G_z$ ,  $c_{x,g}$  is a  $\mathcal{G}$ -loop based at  $x$ . Set

$$(2.52) \quad r_z : g \in G_z \rightarrow [c_z c_{x,g} c_z^{-1}] \in \Gamma.$$

Then,  $r_z$  is a morphism of groups. By (2.47) and (2.51),

$$(2.53) \quad p^{-1}(U_z) = \coprod_{[a] \in \Gamma / \text{Im}(r_z)} V_{z,a}.$$

Using the fact that  $p^{-1}(U_z)$  is open in  $X$ , we can deduce that  $V_{z,a}$  is open in  $X$ .

Put

$$(2.54) \quad H_z = \ker r_z.$$

By (2.52) and (2.54),  $H_z$  acts on  $\widetilde{V}_{z,a}$  by

$$(2.55) \quad (g, [c]) \in H_z \times \widetilde{V}_{z,a} \rightarrow [c][c_{\widehat{p}([c]),g^{-1}}] \in \widetilde{V}_{z,a}.$$

Then  $\pi_{z,a}$  induces an homeomorphism of topological spaces

$$(2.56) \quad H_z \backslash \widetilde{V}_{z,a} \simeq V_{z,a}.$$

As the  $H_z$ -action on  $\widetilde{V}_{z,a}$  is effective,  $(\widetilde{V}_{z,a}, H_z, \pi_{z,a})$  is an orbifold chart for  $V_{z,a}$ . Moreover, the compatibility of each charts is a consequence of (2.44) and the compatibility of charts in  $\widetilde{\mathcal{U}}$ . Hence,  $\{(\widetilde{V}_{z,a}, H_z, \pi_{z,a})\}_{U_z \in \mathcal{U}, a \in \Gamma}$  forms an orbifold atlas on  $X$ .

As  $\Gamma$  is countable,  $X$  is second countable. We will show that  $X$  is Hausdorff. Indeed, take  $y_1, y_2 \in X$  and  $y_1 \neq y_2$ . If  $p(y_1) \neq p(y_2)$ , as  $Z$  is Hausdorff, take respectively open neighborhoods  $U_1$  and  $U_2$  of  $p(y_1)$  and of  $p(y_2)$  such that  $U_1 \cap U_2 = \emptyset$ . Then  $p^{-1}(U_1) \cap p^{-1}(U_2) = \emptyset$ . Assume  $p(y_1) = p(y_2)$ . By adding charts in  $\mathcal{U}$ , we can assume that there is  $U_z \in \mathcal{U}$  such that  $p(y_1) = p(y_2) = z$  with orbifold charts  $(\widetilde{U}_z, G_z, \pi_z)$  centered at



$x$ . Assume  $y_1, y_2$  are represented by  $\mathcal{G}$ -paths  $c_1$  and  $c_2$  starting at  $x_0$  and ending at  $x$ . For  $i = 1, 2$ , set

$$(2.57) \quad a_i = [c_i][c_{z_0}^{-1}] \in \Gamma.$$

As  $y_1 \neq y_2$ , then  $[a_1] \neq [a_2] \in \Gamma / \text{Im}(r_z)$ . Thus,

$$(2.58) \quad y_1 \in V_{z,a_1}, \quad y_2 \in V_{z,a_2}, \quad V_{z,a_1} \cap V_{z,a_2} = \emptyset.$$

In summary, we have shown that  $X$  is an orbifold.

Note that  $\gamma V_{z,a} = V_{z,\gamma a}$ . By (2.53), the set

$$(2.59) \quad \{\gamma \in \Gamma : \gamma V_{z,a} \cap V_{z,a} \neq \emptyset\} = a \text{Im}(r_z) a^{-1} \subset \Gamma$$

is finite. Then the  $\Gamma$ -action on  $X$  is properly discontinuous. As  $\Gamma$  acts on  $\widehat{\mathcal{G}}_0$ , the  $\Gamma$ -action is smooth.

We claim that (2.49) is homeomorphism of topological space. Indeed, by the construction, (2.49) is injective. It is surjective as  $Z$  is connected. The continuity of the inverse (2.49) is a consequence of (2.53).

The isomorphism of orbifolds between  $\Gamma \backslash X$  and  $Z$  is a consequence of Proposition 2.12 and (2.44), (2.56) and (2.59). The proof of Theorem 2.30 is completed.  $\square$

*Remark 2.31.* By (2.53), (2.56), and by the covering orbifold theory of Thurston [T80, Definition 13.2.2],  $p : X \rightarrow Z$  is a covering orbifold of  $Z$ . Moreover, we can show that for any covering orbifold  $p' : Y \rightarrow Z$ , there exists a covering orbifold  $p'' : X \rightarrow Y$  such that the diagram

$$(2.60) \quad \begin{array}{ccc} X & & \\ \downarrow p & \searrow p'' & \\ & Y & \\ & \swarrow p' & \\ & Z & \end{array}$$

commutes. For this reason,  $X$  is called a universal covering orbifold of  $Z$ . As in the case of the classical covering theory of topological spaces, the universal covering orbifold is unique up to covering isomorphism. Also,  $\Gamma$  is isomorphic to the orbifold deck transformation group of  $X$ .

**2.5. Flat vector bundles and holonomy.** In this section, we still assume that  $Z$  is a connected orbifold. Let  $F$  be a proper flat orbifold vector bundle on  $Z$ . Let  $(\mathcal{U}, \widetilde{\mathcal{U}})$  be an orbifold atlas as in Definition 2.14. Let  $\mathcal{G}$  be the associated groupoid. We fix  $x_0 \in \mathcal{G}$ .

For a  $\mathcal{G}$ -path  $c = (b_1, \dots, b_k; g_0, \dots, g_k)$ , the parallel transport  $\tau_c$  of  $F$  along  $c$  is defined by

$$(2.61) \quad \tau_c = g_{k,*}^F \cdots g_{0,*}^F \in \text{GL}_r(\mathbb{C}).$$

It depends only on the homotopy class of  $c$ . In particular, it defines a representation, called holonomy representation of  $F$ ,

$$(2.62) \quad \rho : \Gamma \rightarrow \text{GL}_r(\mathbb{C}).$$

The isomorphic class of the representation  $\rho$  is independent of the choice of orbifold atlas on  $Z$ , of the local trivialization of  $F$ , and of the choice of  $x_0$ . Moreover, it does not depend on the isomorphic class of  $F$ .

Let  $\text{Hom}(\Gamma, \text{GL}_r(\mathbb{C}))/\sim$  be the set of equivalent classes of complex representations of  $\Gamma$  of dimension  $r$ , and let  $\mathcal{M}_r^{\text{pr}}(Z)$  be the set of isomorphic classes of complex flat orbifold vector bundles of rank  $r$  on  $Z$ . By Proposition 2.22 and Remark 2.23, the map

$$(2.63) \quad \rho \in \text{Hom}(\Gamma, \text{GL}_r(\mathbb{C}))/\sim \rightarrow X_\rho \times \mathbb{C}^r \in \mathcal{M}_r^{\text{pr}}(Z)$$

is well-defined.

**Theorem 2.32.** *The map (2.63) is one-one and onto, whose inverse is given by the holonomy representation (2.62).*

*Proof.* Step 1. The holonomy representation of  $X_\rho \times \mathbb{C}^r$  is isomorphic to  $\rho$ . Assume that the orbifold  $Z$  is covered by  $\{U_z\}$  with simply connected orbifold charts  $\{\tilde{U}_z\}$  centered at  $x \in \tilde{U}_z$  and  $X$  is covered by  $\{V_{z,a}\}$  as (2.53) such that  $p(V_{z,a}) = U_z$ .

Take  $\gamma \in \Gamma$ . Let  $c = (b_1, \dots, b_k; g_0, \dots, g_k)$  be a  $\mathcal{G}$ -loop based at  $x_0$  which represents  $\gamma$ . It is enough to show the parallel transport along  $c$  is  $\rho(\gamma)$ . For  $1 \leq i \leq k$ , take

$$(2.64) \quad x_i = b_i(t_{i-1}).$$

Up to equivalence relation of  $c$  and up to adding charts into the orbifold atlas of  $Z$ , we can assume that there are orbifold charts  $\tilde{U}_{z_i}$  of  $Z$  centered at  $x_i$  such that  $b_i : [t_{i-1}, t_i] \rightarrow \tilde{U}_{z_i}$ . Also, we assume that  $\tilde{U}_{z_0}$  is an orbifold chart of  $Z$  centered at  $x_0$ .

Let  $c_{z_1} = c_{x_0, g_0}$  as in (2.45). For  $2 \leq i \leq k$ , set

$$(2.65) \quad c_{z_i} = (b_1, \dots, b_{i-1}; g_0, \dots, g_{i-1}).$$

By (2.42), for  $1 \leq i \leq k$ ,  $c_{z_i}$  is a  $\mathcal{G}$ -path starting at  $x_0$  and ending at  $x_i$  such that

$$(2.66) \quad [c_{z_i}] \in \tilde{V}_{z_i, 1}.$$

We claim that for  $1 \leq i \leq k$ ,

$$(2.67) \quad V_{z_{i-1}, 1} \cap V_{z_i, 1} \neq \emptyset, \quad V_{z_k, 1} \cap V_{z_0, \gamma} \neq \emptyset.$$

Indeed,  $c'_{z_i} = (b_1, \dots, b_{i-1}; g_0, \dots, g_{i-2}, 1)$  projects to the same element of  $X$  as  $c_{z_i}$ , and  $[c'_{z_i}] \in \tilde{V}_{z_{i-1}, 1}$ . Also,  $c'_{z_{k+1}}$  projects to the same element of  $X$  as  $c$ .

Recall that  $\gamma V_{z_0, 1} = V_{z_0, \gamma}$ . By (2.36), (2.37) and (2.67), for  $1 \leq i \leq k-1$ , we have

$$(2.68) \quad g_{i,*} = 1, \quad g_{k,*} = \rho(\gamma).$$

By (2.61), the parallel transport along  $c$  is  $\rho(\gamma)$ .

Step 2. If  $F$  has holonomy  $\rho$ , then  $F$  is isomorphic to  $X_\rho \times \mathbb{C}^r$ . We will construct the bundle isomorphism. By (2.24) and (2.25), the groupoid of the total space  $\mathcal{F}$  is given by  $\mathcal{G}^F = (\mathcal{G}_0 \times \mathbb{C}^r, \mathcal{G}_1 \times \mathbb{C}^r)$ . Let us construct a universal covering orbifold of  $\mathcal{F}$  by determining its groupoid  $\widehat{\mathcal{G}}^F$ .

Take  $(x_0, 0) \in \mathcal{G}_0^F$ . Let  $(c, v)$  be a  $\mathcal{G}^F$ -path starting at  $(x_0, 0) \in \mathcal{G}_0^F$  and ending at  $(x_1, u) \in \mathcal{G}_0^F$ . Then there is a partition of  $[0, 1]$  given by  $0 = t_0 < \dots < t_k = 1$  such that  $c = (b_1, \dots, b_k; g_0, \dots, g_k)$  as in Definition 2.24. Also,  $v = (v_1, \dots, v_k)$ , where  $v_i : [t_{i-1}, t_i] \rightarrow \mathbb{C}^r$  is a continuous path such that  $v_1(0) = 0$ ,  $g_{k,*}^F v_k(1) = u$  and for  $1 \leq i \leq k-1$ ,

$$(2.69) \quad g_{i,*}^F v_i(t_i) = v_{i+1}(t_i).$$

Put  $w : [0, 1] \rightarrow \mathbf{C}^r$  a continuous path such that for  $t \in [t_{i-1}, t_i]$ ,

$$(2.70) \quad w(t) = g_{0,*}^{F,-1} \cdots g_{i-1,*}^{F,-1} v_i(t).$$

Then,

$$(2.71) \quad w(0) = 0, \quad w(1) = \tau_c^{-1} u.$$

We identify  $(c, v)$  with  $(c, w)$  via (2.70). Then,  $(c, v)$  is homotopic to  $(c', v')$  if and only if  $c, c'$  are homotopic as  $\mathcal{G}$ -path and  $w, w'$  are homotopic as ordinary continuous paths in  $\mathbf{C}^r$ . Since any continuous path  $w : [0, 1] \rightarrow \mathbf{C}^r$  such that  $w(0) = 0$  is homotopic to the path  $t \in [0, 1] \rightarrow tw(1)$ , we have the identification

$$(2.72) \quad [c, v] \in \widehat{\mathcal{G}}_0^F \rightarrow ([c], w(1)) = ([c], \tau_c^{-1} u) \in \widehat{\mathcal{G}}_0 \times \mathbf{C}^r.$$

In particular, we have an isomorphism of groups

$$(2.73) \quad [c] \in \Gamma \rightarrow ([c], 0) \in \pi_1^{\text{orb}}(\mathcal{F}, (x_0, 0)),$$

where 0 is the constant loop at  $0 \in \mathbf{C}^r$ .

In the same way, we identify

$$(2.74) \quad ([c, v], g) \in \widehat{\mathcal{G}}_1^F \rightarrow ([c], g, w(1)) \in \widehat{\mathcal{G}}_1 \times \mathbf{C}^r.$$

Then,  $([c], g, w(1))$  represents an arrow from  $([c], w(1)) \in \widehat{\mathcal{G}}_0^F$  to  $([cc_g], w(1)) \in \widehat{\mathcal{G}}_0^F$ . Therefore, the orbit space of  $\widehat{\mathcal{G}}^F$ , which is also the universal covering orbifold of  $\mathcal{F}$ , is given by  $X \times \mathbf{C}^r$ .

By the identification (2.72), the projection (2.41) is given by

$$(2.75) \quad \widehat{p}_\rho : ([c], w(1)) \in \widehat{\mathcal{G}}_0 \times \mathbf{C}^r \rightarrow (\widehat{p}([c]), w(1)) \in \mathcal{G}_0 \times \mathbf{C}^r.$$

The group  $\Gamma$  acts on the left on  $\widehat{\mathcal{G}}_0$ , and on the left on  $\mathbf{C}^r$  by  $\rho$ . As in (2.41), the projection (2.75) is a Galois covering with deck transformation group  $\Gamma$ . And  $\widehat{p}_\rho$  descends to a  $\Gamma$ -invariant continuous map

$$(2.76) \quad p_\rho : X \times \mathbf{C}^r \rightarrow \mathcal{F}.$$

By Theorem 2.30,  $p_\rho$  induces an isomorphism of orbifolds

$$(2.77) \quad X \times_{\rho} \mathbf{C}^r \simeq \mathcal{F}.$$

Using the fact that (2.75) is linear on the  $\mathbf{C}^r$ , we can deduce that (2.77) is an isomorphism of orbifold vector bundles. The proof of Theorem 2.32 is completed.  $\square$

*Remark 2.33.* The properness condition is necessary. Indeed, Theorem 2.32 implies that the proper flat vector bundle is trivial on the universal cover. Consider a non trivial finite group  $H$  acting effectively on  $\mathbf{C}^r$ . Then  $H \backslash \mathbf{C}^r$  is a non proper orbifold vector bundle over a point. Clearly, it is not trivial.

*Remark 2.34.* By (2.28) and Theorem 2.32, we get Corollary 0.2.

### 3. DIFFERENTIAL CALCULUS ON ORBIFOLDS

The purpose of this section is to extend the usual differential calculus to orbifolds.

This section is organized as follows. In subsections 3.1 and 3.2, we define the differential operators, and introduce the de Rham complex and de Rham cohomology for orbifolds.

In subsection 3.3, we introduce the connection and curvature for orbifold vector bundles. We construct certain characteristic forms using Chern-Weil theory. The Euler form, odd Chern character form, their Chern-Simons classes, and their canonical extensions to  $Z \coprod \Sigma Z$  are constructed in detail.

**3.1. Differential operators on orbifolds.** Let  $Z$  be an orbifold with atlas  $(\mathcal{U}, \tilde{\mathcal{U}})$ . Let  $E_1$  and  $E_2$  be two proper orbifold vector bundles on  $Z$ . Assume that  $E_1$  and  $E_2$  are such that (2.20) holds.

A differential operator  $D$  of order  $p$  is a family  $\{\tilde{D}_U : C^\infty(\tilde{U}, \tilde{E}_{1,U}) \rightarrow C^\infty(\tilde{U}, \tilde{E}_{2,U})\}_{U \in \mathcal{U}}$  of  $G_U$ -invariant differential operators of order  $p$  such that if  $g \in \mathcal{G}_1$  is an arrow from  $x_1 \in \tilde{U}_1$  to  $x_2 \in \tilde{U}_2$ , then near  $x_1$ , we have

$$(3.1) \quad g^* \tilde{D}_{U_2} = \tilde{D}_{U_1}.$$

If each  $\tilde{D}_U$  is elliptic, then  $D$  is called elliptic.

If  $s \in C^\infty(Z, E_1)$  is represented by the family  $\{s_U \in C^\infty(\tilde{U}, \tilde{E}_{1,U})^{G_U}\}$  such that (2.29) holds. By (2.29) and (3.1),  $\{\tilde{D}_U s_U \in C^\infty(\tilde{U}, \tilde{E}_{2,U})^{G_U}\}_{U \in \mathcal{U}}$  defines a section of  $E_2$ , which is denoted by  $Ds$ . Clearly,  $D : C^\infty(Z, E_1) \rightarrow C^\infty(Z, E_2)$  is a linear operator such that

$$(3.2) \quad \text{Supp}(Ds) \subset \text{Supp}(s).$$

**Example 3.1.** A differential operator of order 0 is a section of  $\text{Hom}(E_1, E_2)$ .

**Example 3.2.** If  $V \in C^\infty(Z, TZ)$  is a vector field on  $Z$  defined by the family of  $\{\tilde{V}_U \in C^\infty(\tilde{U}, T\tilde{U})^{G_U}\}_{U \in \mathcal{U}}$  of vector fields on  $\tilde{U}$ , then the Lie derivation  $L_V$  is defined by the family of Lie derivations  $\{L_{\tilde{V}_U}\}_{U \in \mathcal{U}}$  on  $\tilde{U}$ .

**3.2. Differential forms on orbifolds.** The de Rham operator  $d^Z : \Omega^\bullet(Z) \rightarrow \Omega^{\bullet+1}(Z)$  is a first order differential operator defined by the family of de Rham operator  $d^{\tilde{U}} : \Omega^\bullet(\tilde{U}) \rightarrow \Omega^{\bullet+1}(\tilde{U})$  on  $\tilde{U}$ .

Clearly,  $(d^Z)^2 = 0$ . Then,  $(\Omega^\bullet(Z), d^Z)$  forms a complex called the orbifold de Rham complex. We denote by  $H^\bullet(Z)$  the corresponding cohomology. By Satake [Sa56],  $H^\bullet(Z)$  coincides with the singular cohomology of the underlying topological space  $Z$ .

If  $F$  is a flat orbifold vector bundle, then the twist de Rham operator  $d^Z : \Omega^\bullet(Z, F) \rightarrow \Omega^{\bullet+1}(Z, F)$  is defined similarly by the family  $\{d^{\tilde{U}} : \Omega^\bullet(\tilde{U}, \mathbb{C}^r) \rightarrow \Omega^{\bullet+1}(\tilde{U}, \mathbb{C}^r)\}_{U \in \mathcal{U}}$ . Then,  $(\Omega^\bullet(Z, F), d^Z)$  forms a complex, and its cohomology  $H^\bullet(Z, F)$  coincides with the cohomology of the sheaf of locally constant sections of  $F$ .

Let us introduce the integration of differential forms. As  $Z$  is Hausdorff and second countable, there exists a partition of unity subordinate to  $\mathcal{U}$ . That means there is a family  $\{\phi_i \in C_c^\infty(Z)\}_{i \in I}$  of smooth functions on  $Z$  with values in  $[0, 1]$  such that the support  $\text{Supp } \phi_i$  is contained in some  $U_i \in \mathcal{U}$ , the family of compact set  $\{\text{Supp } \phi_i\}_{i \in I}$  is locally finite, and

$$(3.3) \quad \sum_{i \in I} \phi_i = 1.$$

Denote by

$$(3.4) \quad \tilde{\phi}_i = \pi_{U_i}^*(\phi_i) \in C_c^\infty(\tilde{U}_i)^{G_{U_i}}.$$

For  $\alpha \in \Omega_c^m(Z, o(TZ))$ , define

$$(3.5) \quad \int_Z \alpha = \sum_i \frac{1}{|G_{U_i}|} \int_{\tilde{U}_i} \tilde{\phi}_i \tilde{\alpha}_{U_i}.$$

By (3.3) and (3.5), we get:

**Proposition 3.3.** *If  $\alpha \in \Omega_c^m(Z, o(TZ))$ , then  $\alpha$  is integrable on  $Z_{\text{reg}}$  such that*

$$(3.6) \quad \int_Z \alpha = \int_{Z_{\text{reg}}} \alpha.$$

Form (3.6), we see that the definition (3.5) does not depend on the choice of orbifold atlas and the partition of unity. Also, when  $\text{Supp}(\alpha) \subset U$ , we have

$$(3.7) \quad \int_Z \alpha = \frac{1}{|G_U|} \int_{\tilde{U}} \tilde{\alpha}.$$

**Theorem 3.4.** *The following identity holds: for  $\alpha \in \Omega_c(Z, o(TZ))$ ,*

$$(3.8) \quad \int_Z d^Z \alpha = 0.$$

*Proof.* By (3.3), we have

$$(3.9) \quad \int_Z d^Z \alpha = \sum_{i \in I} \int_Z d^Z(\phi_i \alpha).$$

As  $\text{Supp } d^Z(\phi_i \alpha) \subset U_i$ , by (3.7) and the Stokes formula on manifolds, we have

$$(3.10) \quad \int_Z d^Z(\phi_i \alpha) = \frac{1}{|G_{U_i}|} \int_{\tilde{U}_i} d^{\tilde{U}_i}(\tilde{\phi}_i \tilde{\alpha}) = 0.$$

Form (3.9) and (3.10), we get (3.8). The proof of Theorem 3.4 is completed.  $\square$

**3.3. Characteristic forms on orbifolds.** In this subsection, we assume all the vector bundle are proper. By (2.28), all the constructions in this subsection extend trivially to non proper orbifold vector bundles.

A metric  $g^E$  on  $E$  is a section in  $C^\infty(Z, \text{End}(E))$  such that  $g^E$  is represented by a family  $\{g^{\tilde{E}_U}\}_{U \in \tilde{U}}$  of  $G_U$ -invariant metrics on  $\tilde{E}_U$  such that (2.29) holds. A connection  $\nabla^E$  on  $E$  is a first order differential operator from  $C^\infty(Z, E)$  to  $\Omega^1(Z, E)$  such that  $\nabla^E$  is represented by a family  $\{\nabla^{\tilde{E}_U}\}_{U \in \tilde{U}}$  of  $G_U$ -invariant connections on  $\tilde{E}_U$  such that (3.1) holds. The curvature  $R^E = (\nabla^E)^2$  is defined as usual. It is a section of  $\Lambda^2(TZ) \otimes_{\mathbf{R}} \text{End}(E)$ .

As in case of manifolds, the metric and connection of  $E$  exist all the time by an argument using the partition of the unity. Moreover,  $\nabla^E$  induces a connection on the tensor bundle of  $E$ . As usual,  $\nabla^E$  is called metric with respect to  $g^E$  if  $\nabla^E g^E = 0$ .

A connection is called flat, if it has vanishing curvature. If  $F$  is a flat orbifold vector bundle, the restriction of the de Rham operator  $d^Z|_{C^\infty(Z, F)} : C^\infty(Z, F) \rightarrow \Omega^1(Z, F)$  defines a flat connection  $\nabla^F$ . On the other hand, if  $F$  admits a flat connection  $\nabla^F$ , then  $F$  is flat. We say  $F$  is unitary flat, if there exists a Hermitian metric  $g^F$  on  $F$  such that  $\nabla^F g^F = 0$ . Clearly, this is equivalent to say the holonomy representation  $\rho$  is unitary.

When  $TZ$  is equipped a Euclidean metric  $g^{TZ}$ , we call  $(Z, g^{TZ})$  the Riemannian orbifold. If  $g^{TZ}$  is defined by the family  $\{g^{T\tilde{U}}\}_{U \in \mathcal{U}}$  of Riemannian metrics, then the family of Levi-civita connections on  $(\tilde{U}, g^{T\tilde{U}})$  defines the Levi-civita connection  $\nabla^{TZ}$  on  $(Z, g^{TZ})$ .

Assume now  $E$  is a real Euclidean orbifold vector bundle of rank  $r$  with a Euclidean metric  $g^E$ , with a metric connection  $\nabla^E$ . The Euler form  $e(E, \nabla^E) \in \Omega^r(Z, o(E))$  is defined by the family of closed forms

$$(3.11) \quad \left\{ e(\tilde{E}_U, \nabla^{\tilde{E}_U}) \in \Omega^r(\tilde{U}, o(\tilde{E}_U)) \right\}_{U \in \mathcal{U}}.$$

Following [Sa57, Section 3.3], if  $Z$  is compact, set

$$(3.12) \quad \chi_{\text{orb}}(Z) = \int_Z e(TZ, \nabla^{TZ}).$$

If  $\nabla^{E'}$  is another metric connection, the class of Chern-Simons form  $\tilde{e}(E, \nabla^E, \nabla^{E'}) \in \Omega^{r-1}(Z, o(E))/d\Omega^{r-2}(Z, o(E))$  is defined by the family

$$(3.13) \quad \left\{ \tilde{e}(\tilde{E}_U, \nabla^{\tilde{E}_U}, \nabla'^{\tilde{E}_U}) \in \Omega^{r-1}(\tilde{U}, o(\tilde{E}_U))/d\Omega^{r-2}(\tilde{U}, o(\tilde{E}_U)) \right\}_{U \in \mathcal{U}}.$$

Clearly, (1.7) still holds true.

Let  $(F, \nabla^F)$  be an orbifold flat vector bundle on  $Z$  with Hermitian metric  $g^F$ . The odd Chern character  $h(\nabla^F, g^F) \in \Omega^{\text{odd}}(Z)$  of  $(F, \nabla^F)$  is defined by the family of closed odd forms

$$(3.14) \quad \left\{ h(\nabla^{\tilde{F}_U}, g^{\tilde{F}_U}) \in \Omega^{\text{odd}}(\tilde{U}) \right\}_{U \in \mathcal{U}}.$$

If  $g'^F$  is another Hermitian metric on  $F$ , the class of Chern-Simons form  $\tilde{h}(\nabla^F, g^F, g'^F) \in \Omega^{\text{even}}(Z)/d\Omega^{\text{odd}}(Z)$  is defined by the family

$$(3.15) \quad \left\{ h(\nabla^{\tilde{F}_U}, g^{\tilde{F}_U}, g'^{\tilde{F}_U}) \in \Omega^{\text{odd}}(\tilde{U})/d\tilde{U}\Omega^{\text{odd}}(\tilde{U}) \right\}_{U \in \mathcal{U}}.$$

As before, (1.23) still holds true.

The degree 1-part of  $h(\nabla^F, g^F)$  and the degree 0-part of  $\tilde{h}(\nabla^F, g^F, g'^F)$  will be especially important in the formulation of Theorem 0.3. We denote by

$$(3.16) \quad \theta(\nabla^F, g^F) = 2h(\nabla^F, g^F)^{[1]}, \quad \tilde{\theta}(\nabla^F, g^F, g'^F) = 2\tilde{h}(\nabla^F, g^F, g'^F)^{[0]}.$$

By (1.20)-(1.23) and (3.16), we have

$$(3.17) \quad \theta(\nabla^F, g^F) = \text{Tr} [g^{F,-1} \nabla^F g^F],$$

and

$$(3.18) \quad d^Z \tilde{\theta}(\nabla^F, g^F, g'^F) = \theta(\nabla^F, g'^F) - \theta(\nabla^F, g^F).$$

The odd Chern character form  $h(\nabla^F, g^F)$  and the Chern-Simons class  $\tilde{h}(\nabla^F, g^F, g'^F)$  can be extended to  $Z \coprod \Sigma Z$ . Recall that for  $U \in \mathcal{U}$  and  $g \in G_U$ ,  $\tilde{U}^g$  is an orbifold charts of  $Z \coprod \Sigma Z$ . The restriction of  $(\tilde{F}_U, \nabla^{\tilde{F}_U})$  to  $\tilde{U}^g$  is a flat vector bundle. The centralizer  $Z_U(g) \subset G_U$  of  $g$  acts fiberwisely on  $\tilde{F}_U$  and perserves  $\nabla^{\tilde{F}_U}$  and  $g^{\tilde{F}_U}$ . The family

$$(3.19) \quad \left\{ h_g(\nabla^{\tilde{F}_U}, g^{\tilde{F}_U}) \in \Omega^{\text{odd}}(\tilde{U}^g) \right\}_{U \in \mathcal{U}, g \in G_U}$$



defines a closed differential form  $h_\Sigma(\nabla^F, g^F) \in \Omega^{\text{odd}}(Z \amalg \Sigma Z)$ . Write  $h_i(\nabla^F, g^F)$  the restriction of  $h_\Sigma(\nabla^F, g^F)$  to  $Z_i \subset Z \amalg \Sigma Z$ . Similarly, we can define

(3.20)

$$\tilde{h}_i(\nabla^F, g^F, g'^F) \in \Omega^{\text{even}}(Z_i)/\Omega^{\text{odd}}(Z_i), \quad \theta_i(\nabla^F, g^F) \in \Omega^1(Z_i), \quad \tilde{\theta}_i(\nabla^F, g^F, g'^F) \in C^\infty(Z_i).$$

As in (2.19), we have

$$(3.21) \quad \begin{aligned} h_0(\nabla^F, g^F) &= h(\nabla^F, g^F), & \tilde{h}_0(\nabla^F, g^F, g'^F) &= \tilde{h}(\nabla^F, g^F, g'^F), \\ \theta_0(\nabla^F, g^F) &= \theta(\nabla^F, g^F), & \tilde{\theta}_0(\nabla^F, g^F, g'^F) &= \tilde{\theta}(\nabla^F, g^F, g'^F). \end{aligned}$$

The rank of  $F$  can be extended to a locally constant function on  $Z \amalg \Sigma Z$  in a similar way. Indeed, the family  $\{\text{Tr}[\rho_U(g)] \in C^\infty(\tilde{U}^g)\}_{U \in \mathcal{U}, g \in G_U}$  of constant functions defines a locally constant function on  $Z \amalg \Sigma Z$ . Denote by  $\rho_i$  its value at  $Z_i$ . Clearly,

$$(3.22) \quad \rho_0 = \text{rk}[F].$$

#### 4. RAY-SINGER METRIC OF ORBIFOLDS

In this section, we assume that  $Z$  is a compact orbifold of dimension  $m$ . Given metrics  $g^{TZ}$  and  $g^F$  on  $TZ$  and  $F$ , we introduce the Ray-Singer metric on the determinant of the de Rham cohomology  $H^*(Z, F)$ . We establish the anomaly formula for the Ray-Singer metric. In particular, when  $Z$  is of odd dimension and orientable, and when  $F$  is unitary flat, the Ray-Singer metric is a topological invariant.

In subsection 4.1, we recall some results about the distributions and Sobolev spaces on compact orbifolds.

In subsection 4.2, we introduce the Hodge Laplacian associated to  $(g^{TZ}, g^F)$ . We extend the classical Hodge theorem and Gauss-Bonnet-Chern Theorem to compact orbifolds.

In subsection 4.3, we construct the analytic torsion and the Ray-Singer metric. Following [BL95], we interpret the analytic torsion as a transgression of odd Chern forms. We then show Theorem 0.3.

**4.1. Distribution and Sobolev space on orbifolds.** For an open subset  $\tilde{U} \subset \mathbb{R}^m$ , we denote by  $\mathcal{D}'(\tilde{U})$  the space of distributions on  $\tilde{U}$ . If  $q \in \mathbb{R}$ , we denote by  $\mathcal{H}^q(\tilde{U})$  the  $q$ -th Sobolev space. For a Hermitian vector bundle  $\tilde{E}_U$  on  $\tilde{U}$ , equipped with the connection, we use the notations  $\mathcal{D}'(\tilde{U}, \tilde{E}_U)$  and  $\mathcal{H}^q(\tilde{U}, \tilde{E}_U)$  for the distribution space and  $q$ -th Sobolev space with coefficients in  $\tilde{E}_U$ .

Let  $(Z, g^{TZ})$  be a compact Riemannian orbifold of dimension  $m$ . Let  $\nabla^{TZ}$  be the Levi-civita connection on  $TZ$ , and let  $R^{TZ}$  be the corresponding curvature. Let  $(E, g^E)$  be a proper Hermitian orbifold vector bundle with connection  $\nabla^E$ . When necessary, we identify  $E$  with  $\overline{E}^*$  via  $g^E$ .

Let  $dv_Z \in \Omega(Z, o(TZ))$  be the Riemannian volume of  $(Z, g^{TZ})$ , that is locally defined by the Riemannian volume  $dv_{\tilde{U}}$  of  $(\tilde{U}, g_U^{TZ})$ . We define the  $L^2$ -metric on  $C^\infty(Z, E)$  as follows: for  $s_1, s_2 \in C^\infty(Z, E)$ ,

$$(4.1) \quad \langle s_1, s_2 \rangle_{C^\infty(Z, E)} = \int_Z \langle s_1(z), s_2(z) \rangle_E dv_Z.$$

Note that  $\langle s_1(z), s_2(z) \rangle_E$  is considered as a smooth function on  $Z$ . Thus, the integral (4.1) does make sense. Let  $L^2(Z, E)$  be the Hilbert completion of  $C^\infty(Z, E)$  with respect to  $\langle \cdot, \cdot \rangle_{C^\infty(Z, E)}$ . By (3.6) and (4.1), we have

$$(4.2) \quad L^2(Z, E) = L^2(Z_{\text{reg}}, E_{\text{reg}}).$$

Let  $\mathcal{D}'(Z, E)$  be the topological dual of  $C^\infty(Z, E)$ . Then,  $\alpha \in \mathcal{D}'(Z, E)$  can be considered as a family  $\{\alpha_U \in \mathcal{D}'(\tilde{U}, \tilde{E}_U)\}_{U \in \mathcal{U}}$  of  $G_U$ -invariant distributions such that (2.29) holds. As in the case of manifolds, we identify  $\alpha \in C^\infty(Z, E)$  with the distribution  $\langle \alpha, \cdot \rangle_{C^\infty(Z, E)}$ . This gives a continuous embedding

$$(4.3) \quad C^\infty(Z, E) \rightarrow \mathcal{D}'(Z, E).$$

Moreover, any differential operator  $D : C^\infty(Z, E) \rightarrow C^\infty(Z, E)$  extends naturally to  $D : \mathcal{D}'(Z, E) \rightarrow \mathcal{D}'(Z, E)$ . Also, if  $D$  is elliptic, then for any  $s \in \mathcal{D}'(Z, E)$  such that  $Ds \in C^\infty(Z, E)$ , we have

$$(4.4) \quad s \in C^\infty(Z, E).$$

Denote by  $\nabla^{\Lambda^\cdot(T^*Z) \otimes_{\mathbf{R}} E}$  the connection on  $\Lambda^\cdot(T^*Z) \otimes_{\mathbf{R}} E$  induced by  $\nabla^{TZ}$  and  $\nabla^E$ . For  $q \in \mathbf{N}$ , take  $\mathcal{H}^q(Z, E)$  to be the Hilbert completion of  $C^\infty(Z, E)$  under the norm defined by

$$(4.5) \quad \|s\|_q^2 = \sum_{j=0}^q \int_Z \left| (\nabla^{\Lambda^\cdot(T^*Z) \otimes_{\mathbf{R}} E})^j s(z) \right|^2 dv_Z.$$

Let  $\mathcal{H}^{-q}(Z, E)$  be the dual of  $\mathcal{H}^q(Z, E)$ . If  $q \in \mathbf{R}$ , we can define  $\mathcal{H}^q(Z, E)$  by interpolation. As in the case of smooth sections,  $s \in \mathcal{H}^q(Z, E)$  can be represented by the family  $\{s_U \in \mathcal{H}^q(\tilde{U}, \tilde{E}_U)^{G_U}\}_{U \in \mathcal{U}}$  of  $G_U$ -invariant sections such that (2.29) holds.

Using these local descriptions, we have

$$(4.6) \quad \bigcap_{q \in \mathbf{R}} \mathcal{H}^q(Z, E) = C^\infty(Z, E), \quad \bigcup_{q \in \mathbf{R}} \mathcal{H}^q(Z, E) = \mathcal{D}'(Z, E).$$

Moreover, if  $q > q'$ , we have the compact embedding

$$(4.7) \quad \mathcal{H}^q(Z, E) \hookrightarrow \mathcal{H}^{q'}(Z, E),$$

and if  $q \in \mathbf{N}$  and  $q > m/2$ , we have the continuous embedding

$$(4.8) \quad \mathcal{H}^q(Z, E) \hookrightarrow C^{q-[m/2]}(Z, E).$$

Also, Schwartz kernel theorem holds for orbifolds. That means for any continuous linear map  $A : C^\infty(Z, E) \rightarrow \mathcal{D}'(Z, E)$ , there exists a unique  $p \in \mathcal{D}'(Z \times Z, E \boxtimes E^*)$  such that for  $s_1 \in C^\infty(Z, E)$  and  $s_2 \in C^\infty(Z, E^*)$ , we have

$$(4.9) \quad \langle As_1, s_2 \rangle = \langle p, s_1 \otimes s_2 \rangle.$$

If  $p$  is of class  $C^k$  for some  $k \in \mathbf{N}$ , then  $A$  is called integral operator. In this case, there exists a family  $\{\tilde{p}_{U, U'} \in C^k(\tilde{U} \times \tilde{U}', \tilde{E}_U \boxtimes \tilde{E}_{U'}^*)^{G_U \times G_{U'}}\}_{U, U' \in \mathcal{U}}$  of  $G_U \times G_{U'}$ -invariant  $C^k$ -sections such that a similar identity as (2.29) holds. Fix  $U \in \tilde{\mathcal{U}}$ ,  $\{p_{U, U'}\}_{U' \in \mathcal{U}}$  defines a section

$$(4.10) \quad \tilde{p}_U \in C^k(\tilde{U} \times Z, \tilde{E}_U \boxtimes E^*).$$

For  $s \in C^\infty(Z, E)$  and  $x \in \tilde{U}$ , set

$$(4.11) \quad \tilde{A}_U s(x) = \int_{z \in Z} \tilde{p}_U(x, z) s(z) dv_Z = \sum_i \frac{1}{|G_{U_i}|} \int_{x' \in \tilde{U}_i} \tilde{\phi}_i(x') \tilde{p}_{U, U_i}(x, x') \tilde{s}_{U_i}(x') dv_{\tilde{U}_i}.$$

Then the family  $\{\tilde{A}_U s\}_{U \in \mathcal{U}}$  satisfies (2.29), and defines the  $C^k$ -section  $As$ .

Now we give another discription of integral operators with  $C^k$ -kernel. By definition, the  $C^k$ -section  $\tilde{p}_{U, U'}(x, x')$  is bounded on  $\tilde{U} \times \tilde{U}'$ . Then,  $\tilde{p}_{U, U'}(x, x')$  defines a bounded section  $p_{\text{reg}} \in C^k(Z_{\text{reg}} \times Z_{\text{reg}}, E_{\text{reg}} \boxtimes E_{\text{reg}}^*)$  such that for  $s \in C^\infty(Z, E)$  and  $z \in Z_{\text{reg}}$ ,

$$(4.12) \quad As(z) = \int_{z' \in Z_{\text{reg}}} p_{\text{reg}}(z, z') s(z') dv_{Z_{\text{reg}}}.$$

Using (4.2),  $A$  extends uniquely to a bounded operator on  $L^2(Z, E)$ . Moreover, as  $p_{\text{reg}}(z, z')$  is bounded,

$$(4.13) \quad \int_{(z, z') \in Z_{\text{reg}} \times Z_{\text{reg}}} |p_{\text{reg}}(z, z')|^2 dv_{Z_{\text{reg}} \times Z_{\text{reg}}} < \infty.$$

Then  $A$  is of Hilbert-Schmidt class.

Assume that  $A$  is of trace class. Then,

$$(4.14) \quad \text{Tr}[A] = \int_{z \in Z_{\text{reg}}} \text{Tr}^E[p_{\text{reg}}(z, z)] dv_{Z_{\text{reg}}}.$$

Note that the family

$$(4.15) \quad \left\{ \frac{1}{|G_U|} \sum_{g \in G_U} g \tilde{p}_{U, U}(g^{-1}x, x) \in C^k(\tilde{U}, \tilde{E}_U) \right\}_{U \in \mathcal{U}}$$

of  $G_U$ -invariant sections satisfies (2.29). It defines a section in  $C^k(Z, \text{End}(E))$ , which is called the restriction of  $p(z, z')$  to diagonal and is denoted by  $p(z, z)$ . Moreover, the restriction of  $p(z, z)$  to  $Z_{\text{reg}}$  is  $p_{\text{reg}}(z, z)$ . By (3.5), (3.6), (4.14), we have

$$(4.16) \quad \text{Tr}[A] = \int_{z \in Z} \text{Tr}^E[p(z, z)] dv_Z.$$

**4.2. Hodge Laplacian.** Let  $F$  be a proper flat orbifold vector bundle of rank  $r$  with flat connection  $\nabla^F$ . Take a Hermitian metric  $g^F$  on  $F$ .

We apply the construction of subsection 4.1 to the Hermitian orbifold vector bundle  $E = \Lambda^\cdot(T^*Z) \otimes_{\mathbf{R}} F$  with the Hermitian metric induced by  $g^{TZ}$  and  $g^F$ , and with the connection  $\nabla^{\Lambda^\cdot(T^*Z) \otimes_{\mathbf{R}} F}$  induced by  $\nabla^{TZ}$  and  $\nabla^F$ . For simplicity, we write

$$(4.17) \quad \mathcal{H}^q = \mathcal{H}^q(Z, \Lambda^\cdot(T^*Z) \otimes_{\mathbf{R}} F), \quad \mathcal{H} = \mathcal{H}^0.$$

Let  $d^{Z,*}$  be the formal adjoint of  $d^Z$  with respect to  $L^2$ -metric  $\langle \cdot, \cdot \rangle_{\Omega^\cdot(Z, F)}$ . Put

$$(4.18) \quad D^Z = d^Z + d^{Z,*}, \quad \square^Z = D^{Z,2} = [d^Z, d^{Z,*}].$$

Then  $d^{Z,*}$  is a first order differential operator, represented by the family of the formal adjoint  $d^{\tilde{U},*}$  of  $d^{\tilde{U}}$  with respect to the  $L^2$ -metric defined by  $g^{T\tilde{U}}$  and  $g^{\tilde{F}_U}$ . Also,  $\square^Z$  is a formally self-adjoint second order elliptic operator acting on  $\Omega^\cdot(Z, F)$ , which is represented by the family of Hodge Laplacian  $\square^{\tilde{U}}$  acting on  $\Omega^\cdot(\tilde{U}, \tilde{F}_U)$  associated with  $g^{T\tilde{U}}$  and  $g^{\tilde{F}_U}$ .

**Proposition 4.1.** *The operator  $(\square^Z, \Omega^\cdot(Z, F))$  is essentially self-adjoint. The domain of the self-adjoint extension is  $\mathcal{H}^2$ .*

*Proof.* Let  $\mathcal{D}_{\min}$  be the Hilbert completion of  $\Omega^\cdot(Z, F)$  with respect to the norm

$$(4.19) \quad \|u\|_{\mathcal{D}_{\min}}^2 = \|u\|_{\mathcal{H}}^2 + \|\square^Z u\|_{\mathcal{H}}^2.$$

Set

$$(4.20) \quad \mathcal{D}_{\max} = \{s \in \mathcal{H} : \square^Z u \in \mathcal{H}\}.$$

Clearly,  $\mathcal{H}^2 \subset \mathcal{D}_{\min} \subset \mathcal{D}_{\max}$ . It is enough to show

$$(4.21) \quad \mathcal{H}^2 = \mathcal{D}_{\max}.$$

Take  $s \in \mathcal{D}_{\max}$ . Assume that  $s$  is represented by a family of  $G_U$ -invariant section  $\{\tilde{s}_U\}_{U \in \mathcal{U}}$  such that (2.29) holds. By (4.20),  $\tilde{s}_U$  and  $\square^{\tilde{U}} \tilde{s}_U$  are of class  $L^2$  on  $\tilde{U}$ . By elliptic regularity, for any  $\phi \in C_c^\infty(\tilde{U})$ ,

$$(4.22) \quad \phi \tilde{s}_U \in \mathcal{H}_c^2(\tilde{U}, \Lambda^\cdot(T^* \tilde{U}) \otimes_{\mathbf{R}} \tilde{F}_U).$$

We consider the embedding

$$(4.23) \quad \mathcal{H}_c^2(\tilde{U}, \Lambda^\cdot(T^* \tilde{U}) \otimes_{\mathbf{R}} \tilde{F}_U)^{G_U} \simeq \mathcal{H}_c^2(U, \Lambda^\cdot(T^* U) \otimes_{\mathbf{R}} F) \hookrightarrow \mathcal{H}^2.$$

Using the partition of the unity (3.3), we have

$$(4.24) \quad s = \sum_i \tilde{\phi}_i \tilde{s}_{U_i}.$$

By (4.22) and (4.24),  $s \in \mathcal{H}^2$ . The proof of Proposition 4.1 is completed.  $\square$

In the sequel, we still denote by  $\square^Z$  the self-adjoint extension of  $(\square^Z, \Omega^\cdot(Z, F))$ . As  $\square^Z$  is non negative, by Proposition 4.1, the operator  $1 + \square^Z : \mathcal{H}^2 \rightarrow \mathcal{H}$  is bounded, invertible and has bounded inverse. By (4.7),  $\square^Z$  has compact resolvent, so it has discrete spectrum. Moreover, for  $k \in \mathbf{N}$ ,  $(1 + \square^Z)^k : \mathcal{H}_q \rightarrow \mathcal{H}_{q-2k}$  is an isomorphism of Hilbert spaces. By (4.8), for  $k \gg 1$ , the operator  $(1 + \square^Z)^{-k}$  has a continuous kernel. In particular,  $(1 + \square^Z)^{-k}$  is of Hilbert-Schmidt class, and  $(1 + \square^Z)^{-2k}$  is of trace class. By the above argument, if  $f$  lies in the Schwartz space  $\mathcal{S}(\mathbf{R})$ , then  $f(\square^Z)$  has a smooth kernel, and is of trace class. For  $t > 0$ , the same statement holds true for the heat operator  $\exp(-t\square^Z)$  of  $\square^Z$ . In this way, most of results on compact manifolds, which have been obtained by the functional calculus of the Hodge Laplacian, still hold true for compact orbifolds.

The following theorem is well-known (c.f. [Ma05, Proposition 2.2], [DY16, Proposition 2.1]). We include a proof for completeness.

**Theorem 4.2.** *The following orthogonal decomposition holds:*

$$(4.25) \quad \Omega^\cdot(Z, F) = \ker \square^Z \oplus \text{Im}(d^Z|_{\Omega^\cdot(Z, F)}) \oplus \text{Im}(d^{Z,*}|_{\Omega^\cdot(Z, F)}).$$

*In particular, we have the canonical identification of the vector spaces*

$$(4.26) \quad \ker \square^Z \simeq H^\cdot(Z, F).$$

*Proof.* Clearly, the three subspaces on right-hand side of (4.25) is orthogonal. For the existence of the decomposition, let  $P^Z$  be the orthogonal projection of  $\mathcal{H}$  onto  $\ker \square^Z$ . Take  $s \in \Omega^\cdot(Z, F)$ . By ellipticity,  $Pu \in \Omega^\cdot(Z, F)$ . Also,  $\square^{Z,-1}(s - Ps)$  is well-defined and smooth. By (4.18),

$$(4.27) \quad s = Ps + d^Z d^{Z,*} \square^{Z,-1}(s - Ps) + d^{Z,*} d^Z \square^{Z,-1}(s - Ps).$$

From (4.27), we get (4.25).

By (4.25), we have

$$(4.28) \quad \ker (d^Z|_{\Omega^\cdot(Z,F)}) = \ker \square^Z \oplus \operatorname{Im} (d^Z|_{\Omega^\cdot(Z,F)}) .$$

Equation (4.26) is a consequence of (4.28). The proof of Theorem 4.2 is completed.  $\square$

Put

$$(4.29) \quad \chi_{\text{top}}(Z, F) = \sum_{i=0}^m (-1)^i \dim_{\mathbb{C}} H^i(Z, F), \quad \chi'_{\text{top}}(Z, F) = \sum_{i=0}^m (-1)^i i \dim_{\mathbb{C}} H^i(Z, F).$$

Let  $N^{\Lambda^\cdot(T^*Z)}$  be the number operator on  $\Lambda^\cdot(T^*Z)$ . We write  $\operatorname{Tr}_s[\cdot] = \operatorname{Tr} \left[ (-1)^{N^{\Lambda^\cdot(T^*Z)}} \cdot \right]$  for the supertrace. By the classical argument of McKean-Singer formula [McS67], we get:

**Proposition 4.3.** *For  $t > 0$ , the following identity holds:*

$$(4.30) \quad \chi_{\text{top}}(Z, F) = \operatorname{Tr}_s \left[ \exp(-t \square^Z) \right] .$$

Recall that  $\chi_{\text{orb}}(Z)$  and  $\rho_i$  are defined in (3.12) and (3.22).

**Theorem 4.4.** *When  $t \rightarrow 0$ , we have*

$$(4.31) \quad \operatorname{Tr}_s \left[ \exp(-t \square^Z) \right] \rightarrow \sum_{i=0}^{l_0} \rho_i \frac{\chi_{\text{orb}}(Z_i)}{m_i} .$$

In particular,

$$(4.32) \quad \chi_{\text{top}}(Z, F) = \sum_{i=0}^{l_0} \rho_i \frac{\chi_{\text{orb}}(Z_i)}{m_i} .$$

*Proof.* Equation (4.32) is a consequence of (4.30) and (4.31). The proof of (4.31) will be given in subsection 5.1, which is based on an argument of finite propagation speeds for solutions of hyperbolic equations.  $\square$

**4.3. Analytic torsion.** Recall that  $P^Z$  is the orthogonal projection of  $\mathcal{H}$  onto  $\ker \square^Z$ . By the short time asymptotic expansions of the heat trace (c.f. [Ma05, Proposition 2.1], also see Corollary 5.4 for a detailed proof), proceeding as in [Se67] or [BeGeVe04, Proposition 9.35], the function

$$(4.33) \quad \theta(s) = -\frac{1}{\Gamma(s)} \int_0^\infty \operatorname{Tr}_s \left[ N^{\Lambda^\cdot(T^*Z)} \exp(-t \square^Z) (1 - P^Z) \right] t^{s-1} dt .$$

defined on  $s \in \mathbb{C}$  such that  $\operatorname{Re}(s) > m/2$  is holomorphic, and has a meromorphic extension to  $\mathbb{C}$  which is holomorphic at  $s = 0$ .

**Definition 4.5.** The analytic torsion of  $F$  is defined by

$$(4.34) \quad T(F, g^{TZ}, g^F) = \exp(\theta'(0)/2) > 0 .$$

*Remark 4.6.* The formalism of Voros [Vo87] on the regularized determinant of the resolvent of Laplacian extends to orbifolds trivially, as the proof relies only on the short time asymptotic expansions of the heat trace and on the functional calculus. Thus the weighted product of zeta regularized determinants

$$(4.35) \quad \sigma \rightarrow \prod_{i=0}^m \det(\sigma + \square^Z|_{\Omega^i(Z,F)})^{(-1)^i i}$$

is a meromorphic function on  $\mathbf{C}$  such that when  $\sigma \rightarrow 0$ , we have

$$(4.36) \quad \prod_{i=0}^m \det (\sigma + \square^Z |_{\Omega^i(Z, F)})^{(-1)^i} = T(Z, g^{TZ}, g^F)^2 \sigma^{\chi'_{\text{top}}(Z, F)} + \mathcal{O}(\sigma^{\chi'_{\text{top}}(Z, F)+1}).$$

We have a generalization of [RS71, Theorem 2.3].

**Proposition 4.7.** *If  $Z$  is an orientable even dimensional compact orbifold and if  $F$  is a unitarily flat orbifold vector bundle, then for any Riemannian metric  $g^{TZ}$  and any flat Hermitian metric  $g^F$ ,*

$$(4.37) \quad \text{Tr}_s \left[ \left( N^{\Lambda^\cdot(T^*Z)} - \frac{m}{2} \right) \exp(-t \square^Z) \right] = 0.$$

In particular,

$$(4.38) \quad T(F, g^{TZ}, g^F) = 1.$$

*Proof.* Let  $*^Z : \Lambda^\cdot(T^*Z) \otimes_{\mathbf{R}} F \rightarrow \Lambda^{m-\cdot}(T^*Z) \otimes_{\mathbf{R}} o(TZ) \otimes_{\mathbf{R}} \overline{F}^*$  be the Hodge star operator. It is a smooth section on  $Z$  locally defined by  $*^{\tilde{U}} : \Lambda^\cdot(T^*\tilde{U}) \otimes_{\mathbf{R}} \tilde{F}_U \rightarrow \Lambda^{m-\cdot}(T^*\tilde{U}) \otimes_{\mathbf{R}} o(T\tilde{U}) \otimes_{\mathbf{R}} \tilde{F}_U^*$ . As  $Z$  is orientable,  $*^Z$  and the metric  $g^F : F \rightarrow \overline{F}^*$  induce an operator  $\star^Z : \Lambda^\cdot(T^*Z) \otimes_{\mathbf{R}} F \rightarrow \Lambda^{m-\cdot}(T^*Z) \otimes_{\mathbf{R}} F$ . Clearly, we have

$$(4.39) \quad \star^Z \left( N^{\Lambda^\cdot(T^*Z)} - \frac{m}{2} \right) \star^{Z,-1} = - \left( N^{\Lambda^\cdot(T^*Z)} - \frac{m}{2} \right).$$

Since  $g^F$  is flat, we have

$$(4.40) \quad \star^Z \square^Z \star^{Z,-1} = \square^Z.$$

Note that when  $m$  is even,  $\star^Z$  is an even isomorphism of  $\Omega^\cdot(Z, F)$ . Hence, by (4.39) and (4.40), we get (4.37). Equation (4.38) is a consequence of (4.37). The proof of Proposition (4.7) is completed.  $\square$

Set

$$(4.41) \quad \lambda = \bigotimes_{i=0}^m (\det H^i(Z, F))^{(-1)^i}.$$

Then  $\lambda$  is a complex line. Let  $|\cdot|_{\lambda}^{\text{RS},2}$  be the  $L^2$ -metric on  $\lambda$  induced via (4.26).

**Definition 4.8.** The Ray-Singer metric is defined by

$$(4.42) \quad \|\cdot\|_{\lambda}^{\text{RS}} = T(F, g^{TZ}, g^F) |\cdot|_{\lambda}^{\text{RS}}.$$

We are going to show the anomaly formula for  $\|\cdot\|_{\lambda}^{\text{RS},2}$ . It is convenient to interpret the analytic torsion as a transgression of odd Chern forms [BL95]. Let  $(g_s^{TZ}, g_s^F)_{s \in \mathbf{R}}$  be a smooth family of metrics on  $TZ$  and  $F$ .

Recall that  $\pi$  is defined in (1.9), and  $g^{\pi^*TZ}$  and  $g^{\pi^*F}$  are defined respectively in (1.10) and (1.26) with  $E = TZ$ . Consider now a trivial infinite dimensional vector bundle  $W$  on  $\mathbf{R}$  defined by

$$(4.43) \quad \mathbf{R} \times \Omega(Z, F) \rightarrow \mathbf{R}.$$

Put

$$(4.44) \quad A' = d^{\mathbf{R}} + d^Z.$$



Then  $A'$  is a flat superconnection on  $W$ . Let  $g^W$  be a Hermitian metric on  $W$  such that  $g_s^W$  is the  $L^2$ -metric on  $\Omega(Z, F)$  induced by  $(g_s^{TZ}, g_s^F)$ . Let  $A''$  be the adjoint of  $A'$  with respect to  $g^W$ . For  $s \in \mathbf{R}$ , denote by  $d_s^{Z,*}$  the formal adjoint of  $d^Z$  with respect to  $g_s^W$ , and by  $*_s$  the Hodge star operator with respect to  $g_s^{TZ}$ . Thus,

$$(4.45) \quad A'' = d^{\mathbf{R}} + d_s^{Z,*} + ds \wedge \left( g_s^{F,-1} \frac{\partial}{\partial s} g_s^F + *_s^{-1} \frac{\partial}{\partial s} *_s \right).$$

Set

$$(4.46) \quad A = \frac{1}{2}(A'' + A'), \quad B = \frac{1}{2}(A'' - A').$$

Then  $A$  is a superconnection on  $W$ , and  $B$  is a fibrewise first order elliptic differential operator. The curvature of  $A$  is given by

$$(4.47) \quad A^2 = -B^2.$$

It is a fibrewise second order elliptic differential operator.

Following [BL95, Definition 2.7], we introduce a deformation of  $g^W$ . For  $t > 0$ , set

$$(4.48) \quad g_t^W = t^{N^{\Lambda^*(T^*Z)}/2} g^W.$$

Let  $A_t''$  be the adjoint of  $A'$  with respect to  $g_t^W$ . Clearly,

$$(4.49) \quad A_t'' = t^{-N^{\Lambda^*(T^*Z)}} A'' t^{N^{\Lambda^*(T^*Z)}}.$$

We define  $A_t$  and  $B_t$  as in (4.46), i.e.,

$$(4.50) \quad A_t = \frac{1}{2}(A_t'' + A'), \quad B_t = \frac{1}{2}(A_t'' - A').$$

**Theorem 4.9.** *For  $t > 0$ , the following identity of smooth functions of  $s \in \mathbf{R}$  holds:*

$$(4.51) \quad \text{Tr}_s [\exp(B_t^2)] = \chi_{\text{top}}(Z, F).$$

*Proof.* Theorem 4.9 can be proved using the technique of the family local index theory as in [BL95, Theorem 3.5]. As here the parameter space  $\mathbf{R}$  is of dimension 1, we give a short proof. By the construction,  $\text{Tr}_s [\exp(B_t^2)]$  is an even form. Thus, it is a function on  $\mathbf{R}$ . By (4.44), (4.45), (4.49) and (4.50), we have

$$(4.52) \quad \text{Tr}_s [\exp(B_t^2)] = \text{Tr}_s [\exp(-t \square_s^Z / 4)],$$

where  $\square_s^Z$  is the Hodge Laplacian for  $(g_s^{TX}, g_s^F)$ . By (4.30), (4.31) and (4.52), we get (4.51). The proof of Theorem 4.9 is completed.  $\square$

Recall that  $h$  is defined in (1.21). Following [BL95, (2.22) and (2.23)], for  $t > 0$ , set

$$(4.53) \quad u_t = \text{Tr}_s [h(B_t)] \in \Omega^1(\mathbf{R}), \quad v_t = \text{Tr}_s \left[ \frac{N^{\Lambda^*(T^*Z)}}{2} h'(B_t) \right] \in C^\infty(\mathbf{R}).$$

By [BL95, Theorem 1.8],  $u_t$  is a closed 1-form on  $\mathbf{R}$ .

**Proposition 4.10.** *For  $t > 0$ , the following identity of 1-forms holds on  $\mathbf{R}$ :*

$$(4.54) \quad \frac{\partial}{\partial t} u_t = d^{\mathbf{R}} \left( \frac{v_t}{t} \right).$$

*In particular, the cohomology class  $[u_t] \in H^1(\mathbf{R})$  does not depend on  $t > 0$ .*

*Proof.* The proposition can be proved in the same way as [BL95, Theorems 2.9 and 3.20]. Here we indicate the essential step. Consider instead a family of two parameters of metrics  $(g_s^{TZ}/s_1, g_s^F)_{(s,s_1) \in \mathbf{R} \times \mathbf{R}_+^*}$ . Denote by

$$(4.55) \quad \underline{\pi} : \mathbf{R} \times \mathbf{R}_+ \times Z \rightarrow Z.$$

Let  $g^{\pi^*(TZ)}$  and  $g^{\pi^*F}$  be metrics on  $\underline{\pi}^*(TZ)$  and  $\underline{\pi}^*F$  such that for  $(s, s_1) \in \mathbf{R} \times \mathbf{R}_+^*$ ,

$$(4.56) \quad (g^{\pi^*(TZ)}, g^{\pi^*F})|_{(s,s_1)} = (g_s^{TZ}/s_1, g_s^F).$$

Consider now the corresponding infinitesimal vector bundle

$$(4.57) \quad \mathbf{R} \times \mathbf{R}_+ \times \Omega(Z, F) \rightarrow \mathbf{R} \times \mathbf{R}_+.$$

As in (4.44)-(4.46) and with some evident modifications, we can define the objects  $\underline{A}_t$ ,  $\underline{B}_t$  and also the odd Chern form  $\underline{u}_t = \text{Tr}_s[h(\underline{B}_t)]$ . By definition,

$$(4.58) \quad \underline{A}_t = A_{s_1 t} + \left( N^{\Lambda(T^*Z)} - \frac{m}{2} \right) \frac{ds_1}{2s_1}, \quad \underline{B}_t = B_{s_1 t} + \left( N^{\Lambda(T^*Z)} - \frac{m}{2} \right) \frac{ds_1}{2s_1}.$$

Using (4.58), we get

$$(4.59) \quad \underline{u}_t = u_{s_1 t} + \left( v_{s_1 t} - \frac{m}{4} \text{Tr}_s[h'(B_{s_1 t})] \right) \frac{ds_1}{s_1}.$$

Note that Theorem 4.9 still holds true when the test function  $e^{x^2}$  is replaced by  $h'(x)$ . That means

$$(4.60) \quad \text{Tr}_s[h'(B_{s_1 t})] = h'(0)\chi_{\text{top}}(Z, F).$$

As  $d^{\mathbf{R} \times \mathbf{R}_+^*}(\underline{u}_t) = 0$ , by (4.59) and (4.60), we get

$$(4.61) \quad \frac{\partial}{\partial s_1} u_{s_1 t} = d^{\mathbf{R}} \left( \frac{v_{s_1 t}}{s_1} \right).$$

By letting  $t = 1$  in (4.61), we get (4.54). The proof of Proposition 4.10 is completed.  $\square$

For a smooth family  $\{\alpha_t\}_{t>0}$  of differential forms on  $\mathbf{R}$ , we say  $\alpha_t = \mathcal{O}(t)$  if for all the compact  $K \subset \mathbf{R}$ , and for all  $k \in \mathbf{N}$ , there is  $C > 0$  such that  $\|\alpha\|_{C^k(K)} \leq Ct$ .

The subspace  $\ker(\square_s^Z) \subset \Omega(Z, F)$  defines a finite dimensional subbundle  $W_0 \subset W$  on  $\mathbf{R}$ . By Theorem 4.2, the typical fiber of  $W_0$  is  $H^*(Z, F)$ . As in [BL95, Section III.f], we equip  $W_0$  with the restricted metric  $g^{W_0}$  and the induced connection  $\nabla^{W_0}$ .

**Theorem 4.11.** *The following identities of 1-forms hold on  $\mathbf{R}$ : as  $t \rightarrow 0$ ,*

$$(4.62) \quad u_t = \sum_{i=0}^{l_0} \frac{1}{m_i} \int_{Z_i} e(\pi^*(TZ_i), \nabla^{\pi^*(TZ_i)}) h_i(\nabla^{\pi^*F}, g^{\pi^*F}) + \mathcal{O}(\sqrt{t});$$

as  $t \rightarrow \infty$ ,

$$(4.63) \quad u_t = h(\nabla^{W_0}, g^{W_0}) + \mathcal{O}(1/\sqrt{t}).$$

In particular, we have the identity in  $H^1(\mathbf{R})$ ,

$$(4.64) \quad h(\nabla^{W_0}) = \sum_{i=0}^{l_0} \frac{1}{m_i} \int_{Z_i} e(\pi^*(TZ_i)) h_i(\nabla^{\pi^*F}).$$

*Proof.* Equation (4.63) can be proved directly using the argument given in [BL95, Theorem 3.16], which is based on functional calculus. The proof of (4.62) will be given in subsection 5.2.  $\square$

**Corollary 4.12.** *As  $t \rightarrow 0$ ,*

$$(4.65) \quad v_t = \frac{m}{4} \chi_{\text{top}}(Z, F) h'(0) + \mathcal{O}(\sqrt{t}).$$

*As  $t \rightarrow \infty$ ,*

$$(4.66) \quad v_t = \frac{1}{2} \chi'_{\text{top}}(Z, F) h'(0) + \mathcal{O}(1/\sqrt{t}).$$

*Proof.* We follow the proof of [BL95, Theorem 3.21]. Remark that when the parameter space is replaced by  $\mathbf{R} \times \mathbf{R}_+$  as in the proof of Proposition 4.10, and with evident modifications, Theorem 4.11 still holds true for  $\underline{u}_t$ . More precisely, as  $t \rightarrow 0$ ,

$$(4.67) \quad \underline{u}_t = \sum_{i=0}^{l_0} \frac{1}{m_i} \int_{Z_i} e(\nabla^{\pi^*(TZ_i)}, \pi^*(TZ_i)) h_i(\nabla^{\pi^*F}, g^{\pi^*F}) + \mathcal{O}(\sqrt{t});$$

and if  $(\underline{W}_0, \nabla^{\underline{W}_0}, g^{\underline{W}_0})$  is the corresponding objects for  $(W_0, \nabla^{W_0}, g^{W_0})$ , as  $t \rightarrow \infty$ ,

$$(4.68) \quad \underline{u}_t = h(\nabla^{\underline{W}_0}, g^{\underline{W}_0}) + \mathcal{O}(1/\sqrt{t}).$$

By (4.59) and (4.60), we have the identity in  $C^\infty(\mathbf{R})$ ,

$$(4.69) \quad v_t = \underline{u}_t \left( \frac{\partial}{\partial s_1} \right) \Big|_{s_1=1} + \frac{m}{4} \chi_{\text{top}}(Z, F) h'(0).$$

By construction, the term  $e(\nabla^{\pi^*(TZ_i)}, \pi^*(TZ_i))$  and  $h_i(\nabla^{\pi^*F}, g^{\pi^*F})$  do not contain the variable  $ds_1$ . By (4.67), as  $t \rightarrow 0$ ,

$$(4.70) \quad \underline{u}_t \left( \frac{\partial}{\partial s_1} \right) \Big|_{s_1=1} = \mathcal{O}(\sqrt{t}).$$

By (4.69) and (4.70), we get (4.65).

By (1.20), we have

$$(4.71) \quad \omega(\nabla^{\underline{W}_0}, g^{\underline{W}_0}) = \omega(\nabla^{W_0}, g^{W_0}) + \frac{ds_1}{s_1} \left( N^{\Lambda^*(T^*Z)} - \frac{m}{2} \right).$$

By (3.14), (4.68) and (4.71), as  $t \rightarrow \infty$ , we get

$$(4.72) \quad \underline{u}_t \left( \frac{\partial}{\partial s_1} \right) \Big|_{s_1=1} \rightarrow \frac{1}{2} \chi'_{\text{top}}(Z, F) h'(0) - \frac{m}{4} \chi_{\text{top}}(Z, F) h'(0) + \mathcal{O}(1/\sqrt{t}).$$

By (4.69) and (4.72), we get (4.66). The proof of Corollary 4.12 is completed.  $\square$

Recall that  $s \in \mathbf{R} \rightarrow \log T(F, g_s^{TZ}, g_s^F)$  is a smooth function. By Corollary 4.12, the integration

$$(4.73) \quad \int_0^\infty \left\{ v_t - \frac{1}{2} \chi'_{\text{top}}(Z, F) h'(0) - \left( \frac{4}{n} \chi_{\text{top}}(Z, F) - \frac{1}{2} \chi'_{\text{top}}(Z, F) \right) h' \left( \frac{i\sqrt{t}}{2} \right) \right\} \frac{dt}{t}$$

converges. Proceeding as [BL95, Theorem 3.29], we can show that the following identity of smooth functions holds on  $\mathbf{R}$ ,

$$(4.74) \quad \begin{aligned} & \log T(F, g^{TZ}, g^F) \\ &= - \int_0^\infty \left\{ v_t - \frac{1}{2} \chi'_{\text{top}}(Z, F) h'(0) - \left( \frac{4}{n} \chi_{\text{top}}(Z, F) - \frac{1}{2} \chi'_{\text{top}}(Z, F) \right) h' \left( \frac{i\sqrt{t}}{2} \right) \right\} \frac{dt}{t} \end{aligned}$$

By Proposition 4.10, Theorem 4.11 and Corollary 4.12, we can refine (4.64) as an identity in  $\Omega^1(\mathbf{R})$ ,

$$(4.75) \quad d^{\mathbf{R}}(\log T(F, g^{TZ}, g^F)) \\ = \sum_{i=1}^{l_0} \frac{1}{m_i} \int_{Z_i} e(\pi^*(TZ_i), \nabla^{\pi(TZ_i)}) h_i(\nabla^{\pi^*F}, g^{\pi^*F}) - h(\nabla^{W_0}, g^{W_0}).$$

Let  $(g^{TX}, g^F)$  and  $(g'^{TX}, g'^F)$  be two pairs of metrics on  $TX$  and  $F$ . Let  $\|\cdot\|_{\lambda}^{\text{RS},2}$  and  $\|\cdot\|_{\lambda}^{\text{RS},2}$  be the corresponding Ray-Singer metrics on  $\lambda$ . We restate Theorem 0.3.

**Theorem 4.13.** *The following identity holds:*

$$(4.76) \quad \log \left( \frac{\|\cdot\|_{\lambda}^{\text{RS},2}}{\|\cdot\|_{\lambda}^{\text{RS},2}} \right) = \sum_{i=0}^{l_0} \frac{1}{m_i} \int_{Z_i} \left( \tilde{\theta}_i(\nabla^F, g^F, g'^F) e(TZ_i, \nabla^{TZ_i}) \right. \\ \left. - \theta_i(\nabla^F, g^F) \tilde{e}(TZ_i, \nabla^{TZ_i}, \nabla'^{TZ_i}) \right).$$

*Proof.* Take a smooth family of metrics  $(g_s^{TZ}, g_s^F)_{s \in \mathbf{R}}$  such that

$$(4.77) \quad (g_s^{TZ}, g_s^F)|_{s=0} = (g^{TZ}, g^F), \quad (g_s^{TZ}, g_s^F)|_{s=1} = (g'^{TZ}, g'^F).$$

If  $\|\cdot\|_{\lambda,s}^{\text{RS},2}$  is the Ray-Singer metric for  $(g_s^{TZ}, g_s^F)$ , by (4.42), and by comparing the degrees of each side of (4.75), we have

$$(4.78) \quad ds \wedge \frac{\partial}{\partial s} \left\{ \log \left( \|\cdot\|_{\lambda,s}^{\text{RS},2} \right) \right\} = \sum_{i=0}^{l_0} \frac{1}{m_i} \int_{Z_i} e(\pi^*(TZ_i), \nabla^{\pi^*(TZ_i)}) \theta_i(\nabla^{\pi^*F}, g^{\pi^*F}).$$

By (1.12) and (1.27) with  $\alpha_{i,s} \in \Omega^{\dim Z_i - 1}(Z_i, o(TZ_i))$  and  $\beta_{i,s}^{[0]} \in C^\infty(Z_i)$  defined in an obvious way, we have

$$(4.79) \quad e(\pi^*(TZ_i), \nabla^{\pi^*(TZ_i)}) = e(TZ_i, \nabla^{TZ_i}) + d^Z \int_0^s \alpha_{i,s} ds + ds \wedge \alpha_{i,s}, \\ \theta_i(\nabla^{\pi^*F}, g^{\pi^*F}) = \theta_i(\nabla^F, g^F) + 2d^Z \int_0^s \beta_{i,s}^{[0]} ds + 2ds \wedge \beta_{i,s}^{[0]}.$$

By (4.79), we have

$$(4.80) \quad \int_{Z_i} e(\pi^*(TZ_i), \nabla^{\pi^*(TZ_i)}) \theta_i(\nabla^{\pi^*F}, g^{\pi^*F}) \\ = ds \left( 2 \int_{Z_i} \beta_{i,s}^{[0]} e(TZ_i, \nabla^{\pi^*(TZ_i)}) - \theta_i(\nabla^F, g^F) \wedge \alpha_{i,s} \right).$$

By integrating (4.80) with respect to the variable  $s$  from 0 to 1, and by (4.78), we get (4.76). The proof of Theorem 4.13 is completed.  $\square$

**Corollary 4.14.** *If all the  $Z_i$ 's are of odd dimension, then  $\|\cdot\|_{\lambda}^{\text{RS},2}$  does not depend on  $g^{TZ}$  or  $g^F$ . In particular, this is the case if  $Z$  is an orientable odd dimensional orbifold.*

*Proof.* When  $\dim Z_i$  is odd,  $e(TZ_i, \nabla^{TZ_i}) = 0$  and  $e(TZ_i, \nabla^{TZ_i}, \nabla'^{TZ_i}) = 0$ . By Theorem 4.13, we get Corollary 4.14.  $\square$

**Corollary 4.15.** *If for  $0 \leq i \leq l$ ,*

$$(4.81) \quad \chi_{\text{orb}}(Z_i) = 0,$$

*and if  $F$  is unitarily flat, then  $\|\cdot\|_{\lambda}^{\text{RS},2}$  does not depend on  $g^{TZ}$  or on the flat Hermitian metric  $g^F$ .*

*Proof.* Take  $(g^{TX}, g^F)$  and  $(g'^{TX}, g'^F)$  two pairs of metrics on  $TX$  and  $F$  such that  $\nabla^F g^F = 0$  and  $\nabla^F g'^F = 0$ . Then, for  $0 \leq i \leq l$ ,

$$(4.82) \quad \theta_i(\nabla^F, g^F) = \theta_i(\nabla^F, g'^F) = 0.$$

By (3.18) and (4.82),  $\tilde{\theta}_i(\nabla^F, g^F, g'^F)$  is closed. It becomes a constant  $c_i \in \mathbb{C}$  as  $Z_i$  is connected. Using (4.81), we get

$$(4.83) \quad \int_{Z_i} \tilde{\theta}_i(F, g^F, g'^F) e(TZ_i, \nabla^{TZ_i}) = c_i \int_{Z_i} e(TZ_i, \nabla^{TZ_i}) = 0.$$

By (4.76), (4.82) and (4.83), we get  $\|\cdot\|_{\lambda}^{\text{RS},2} = \|\cdot\|_{\lambda}'^{\text{RS},2}$ . The proof of Corollary 4.15 is completed.  $\square$

*Remark 4.16.* If  $F$  is not proper, we can define the analytic torsion and Ray-Singer metric in the same way. We have

$$(4.84) \quad T(F, g^{TZ}, g^F) = T(F^{\text{pr}}, g^{TZ}, g^{F^{\text{pr}}}), \quad H^*(X, F) = H^*(X, F^{\text{pr}}).$$

Also, the Ray-Singer metrics of  $F$  and  $F^{\text{pr}}$  coincides. For this reason, all the result in this Section holds true for non proper flat orbifold vector bundle.

## 5. ESTIMATES ON HEAT KERNELS

The purpose of this section is to prove Theorems 4.4 and 4.11 in a unified way. Using an argument due to [Ma05, P. 2230] (c.f. also in [MaMar07]), which is based on the finite propagation speeds for the solutions of hyperbolic equations [ChP81, Section 7.8], we can turn our problem into a local one. As the orbifold locally is a quotient of a manifold by some finite group, we can then rely on the results of Bismut-Goette [BG01], where the authors there consider some similar problems in the equivariant setting.

This section is organized as follows. In subsection 5.1, we explain how to reduce the analysis on orbifolds to manifolds. We show Theorem 4.4.

In subsections 5.2, we show Theorem 4.11 in the same way.

**5.1. Proof of Theorem 4.4.** We follow [BG01, Section 13.2]. If  $a \in \mathbb{C}$ , we have

$$(5.1) \quad \exp(-a^2) = \int_{\mathbb{R}} e^{2isa} e^{-s^2} \frac{ds}{\sqrt{\pi}}.$$

Take  $\alpha_0 > 0$ . Let  $f : \mathbb{R} \rightarrow [0, 1]$  be a smooth even function such that

$$(5.2) \quad f(s) = \begin{cases} 1, & |s| \leq \alpha_0/2; \\ 0, & |s| \geq \alpha_0. \end{cases}$$

Put

$$(5.3) \quad g(s) = 1 - f(s).$$

**Definition 5.1.** For  $t > 0$  and  $a \in \mathbb{C}$ , set

$$(5.4) \quad F_t(a) = \int_{\mathbb{R}} e^{2isa} e^{-s^2} f(\sqrt{t}s) \frac{ds}{\sqrt{\pi}}, \quad G_t(a) = \int_{\mathbb{R}} e^{2isa} e^{-s^2} g(\sqrt{t}s) \frac{ds}{\sqrt{\pi}}.$$

By (5.1), (5.3) and (5.4), we get

$$(5.5) \quad \exp(-a^2) = F_t(a) + G_t(a).$$

Moreover,  $F_t$  and  $G_t$  are even holomorphic functions, whose restriction to  $\mathbf{R}$  lies in  $\mathcal{S}(\mathbf{R})$ . By (5.4), we find that given  $m, m' \in \mathbf{N}$ ,  $c > 0$ , there exist  $C > 0, C' > 0$  such that if  $t \in (0, 1]$ ,  $a \in \mathbf{C}$ ,  $|\operatorname{Im}(a)| \leq c$ ,

$$(5.6) \quad |a|^m \left| G_t^{(m')}(a) \right| \leq C \exp(-C'/t).$$

There exist uniquely well-defined holomorphic functions  $\mathcal{F}_t(a)$  and  $\mathcal{G}_t(a)$  such that

$$(5.7) \quad F_t(a) = \mathcal{F}_t(a^2), \quad G_t(a) = \mathcal{G}_t(a^2).$$

By (5.5) and (5.7), we have

$$(5.8) \quad \exp(-a) = \mathcal{F}_t(a) + \mathcal{G}_t(a).$$

By (5.8), we get

$$(5.9) \quad \exp(-t\Box^Z) = \mathcal{F}_t(t\Box^Z) + \mathcal{G}_t(t\Box^Z).$$

If  $A$  is a bounded operator, let  $\|A\|$  be its norm as bounded operator. If  $A$  is of trace class, let  $\|A\|_1$  be its the norm as trace class operator.

**Proposition 5.2.** *There exist  $c > 0$  and  $C > 0$  such that for  $t \in (0, 1]$ ,*

$$(5.10) \quad \|\mathcal{G}_t(t\Box^Z)\|_1 \leq Ce^{-c/t}.$$

In particular, as  $t \rightarrow 0$ , we have

$$(5.11) \quad \operatorname{Tr}_s [\exp(-t\Box^Z)] = \operatorname{Tr}_s [\mathcal{F}_t(t\Box^Z)] + \mathcal{O}(e^{-c/t}).$$

*Proof.* By (5.6) and (5.7), for any  $k \in \mathbf{N}$ , the operator  $(1 + \Box^Z)^k \mathcal{G}_t(t\Box^Z)$  is bounded on  $\mathcal{H}$  such that there exist  $C > 0$  and  $C' > 0$ ,

$$(5.12) \quad \|(1 + \Box^Z)^k \mathcal{G}_t(t\Box^Z)\| \leq C \exp(-C'/t).$$

Take  $k \in \mathbf{N}$  big enough such that  $(1 + \Box^Z)^{-k}$  is of trace class. Then

$$(5.13) \quad \|\mathcal{G}_t(t\Box^Z)\|_1 \leq \|(1 + \Box^Z)^{-k}\|_1 \|(1 + \Box^Z)^k \mathcal{G}_t(t\Box^Z)\|.$$

By (5.12) and (5.13), we get (5.10). The proof of Proposition 5.2 is completed.  $\square$

Assume that  $Z$  is covered by a finite family  $\mathcal{U} = \{U_i\}_{i \in I}$  of connected open sets with orbifold atlas  $\tilde{\mathcal{U}} = \{(\tilde{U}_i, G_{U_i}, \pi_{U_i})\}_{i \in I}$ . Let  $\{\phi_i\}_{i \in I}$  be a partition of unity subordinate to  $\{U_i\}_{i \in I}$ . Let  $\mathcal{F}_t(t\Box^Z)(z, z')$  be the smooth kernel of  $\mathcal{F}_t(t\Box^Z)$  with respect to  $dv_Z$ , which is represented by the family  $\left\{ \widetilde{\mathcal{F}_t(t\Box^Z)}_{U, U'}(x, x') \right\}_{U, U' \in \mathcal{U}}$  of  $G_U \times G_{U'}$ -invariant sections. By (3.5), (4.15) and (4.16), we have

$$(5.14) \quad \operatorname{Tr}_s [\mathcal{F}_t(t\Box^Z)] = \sum_{i \in I} \frac{1}{|G_{U_i}|} \sum_{g \in G_{U_i}} \int_{\tilde{U}_i} \tilde{\phi}_i(x) \operatorname{Tr}_s \left[ g \widetilde{\mathcal{F}_t(t\Box^Z)}_{U_i, U_i}(g^{-1}x, x) \right] dv_{\tilde{U}_i}.$$

For  $i \in I$ , and  $g \in G_{U_i}$ , let  $N_{\tilde{U}_i^g/\tilde{U}_i}$  be the normal bundle of  $\tilde{U}_i^g$  in  $\tilde{U}_i$ . We identify  $N_{\tilde{U}_i^g/\tilde{U}_i}$  with the orthogonal bundle of  $T\tilde{U}_i^g$  in  $T\tilde{U}_i|_{\tilde{U}_i^g}$ . For  $\epsilon_0 > 0$ , set

$$(5.15) \quad N_{\tilde{U}_i^g/\tilde{U}_i, \epsilon_0} = \left\{ (y, Y) \in N_{\tilde{U}_i^g/\tilde{U}_i} : \operatorname{dist}(y, \operatorname{Supp}(\tilde{\phi}_i)) < \epsilon_0, |Y| < \epsilon_0 \right\}.$$



Take  $\epsilon_0 > 0$  small enough such that for all  $i \in I$ ,  $\overline{\{x \in \tilde{U}_i : \text{dist}(x, \text{Supp}(\tilde{\phi}_i)) < \epsilon_0\}} \subset \tilde{U}_i$ , and such that all  $i \in I$ ,  $g \in G_{U_i}$ , the exponential map  $(y, Y) \in N_{\tilde{U}_i^g/\tilde{U}_i, \epsilon_0} \rightarrow \exp_y(Y) \in \tilde{U}_i$  defines a diffeomorphism from  $N_{\tilde{U}_i^g/\tilde{U}_i, \epsilon_0}$  onto its image  $\tilde{U}_{i, \epsilon_0, g} \subset \tilde{U}_i$ . Also, there exists  $\delta_0 > 0$  such that for all  $i \in I$ ,  $g \in G_{U_i}$  if  $x \in \text{Supp}(\tilde{\phi}_i)$  and  $\text{dist}(g^{-1}x, x) < \delta_0$ , then

$$(5.16) \quad x \in \tilde{U}_{i, \epsilon_0, g}.$$

Let  $dv_{\tilde{U}_i^g}$  be the induced Riemannian volume of  $\tilde{U}_i^g$ , and let  $dY$  be the induced Lebesgue volume on the fiber of  $N_{\tilde{U}_i^g/\tilde{U}_i}$ . Let  $k_i : N_{\tilde{U}_i^g/\tilde{U}_i, \epsilon_0} \rightarrow \mathbf{R}_+^*$  be a smooth function such that on  $\tilde{U}_{i, \epsilon_0, g}$  we have

$$(5.17) \quad dv_{\tilde{U}_i} = k_i(y, Y) dv_{\tilde{U}_i^g} dY.$$

Clearly,  $k_i(y, 0) = 1$ .

For  $x \in \text{Supp}(\tilde{\phi}_i)$  and  $r \in (0, \epsilon_0)$ , let  $B_x^{\tilde{U}_i}(r)$  be the geodesic ball of center  $x$  and radius  $r$ . As (4.10),  $\tilde{\mathcal{F}}_t(t\Box^Z)_{U_i}(x, z)$  is a smooth section on  $\tilde{U}_i \times Z$ . Using the result of the finite propagation speeds for the solutions of hyperbolic equations [ChP81, Section 7.8], by taking  $\alpha_0 < \frac{1}{4} \min\{\delta_0, \epsilon_0\}$ , for  $x \in \text{Supp}(\tilde{\phi}_i)$ , we find the support of  $\tilde{\mathcal{F}}_t(t\Box^Z)_{U_i}(x, \cdot)$  in  $U_i$ , and the support of  $\tilde{\mathcal{F}}_t(t\Box^Z)_{U_i, U_i}(x, \cdot)$  in  $B_x^{\tilde{U}_i}(4\alpha_0)$ . Moreover,  $\tilde{\mathcal{F}}_t(t\Box^Z)_{U_i, U_i}(x, \cdot)$  depends only on the Hodge Laplacian  $\Box^{\tilde{U}_i}$  acting on  $\Omega(\tilde{U}, \tilde{F}_U)$ . Using (5.16) and (5.17), we get

$$(5.18) \quad \int_{\tilde{U}_i} \tilde{\phi}_i(x) \text{Tr}_s \left[ g \tilde{\mathcal{F}}_t(t\Box^Z)_{U_i, U_i}(g^{-1}x, x) \right] dv_Z \\ = \int_{y \in \tilde{U}_i^g} dv_{\tilde{U}_i^g} \int_{Y \in N_{\tilde{U}_i^g/\tilde{U}_i, y}, |Y| < \epsilon_0} \phi_i(y, Y) \text{Tr}_s \left[ g \tilde{\mathcal{F}}_t(t\Box^Z)_{U_i, U_i}(g^{-1}(y, Y), (y, Y)) \right] k_i(y, Y) dY.$$

Consider an isometric embedding of  $(\tilde{U}_i, g^{T\tilde{U}_i})$  into a compact manifold  $(X_i, g^{TX_i})$ . We extend the trivial Hermitian vector bundle  $(\tilde{F}_i, g^{\tilde{F}_{U_i}})$  to a trivial Hermitian vector bundle  $(F_i, g^{F_i})$  on  $X_i$ . Thus, when restricted on  $\tilde{U}_i$ ,

$$(5.19) \quad \Box^{\tilde{U}_i} = \Box^{X_i}.$$

Using the argument of the finite propagation speeds for the solutions of hyperbolic equations again, for  $x, y \in \tilde{U}_i$ , we have

$$(5.20) \quad \tilde{\phi}_i(x) \tilde{\mathcal{F}}_t(t\Box^Z)_{U_i, U_i}(x, y) = \tilde{\phi}_i(x) \mathcal{F}_t(t\Box^{X_i})(x, y).$$

Recall that for  $x \in \tilde{U}_i$  and  $g \in G_{U_i}$ ,  $g : F_{g^{-1}x} \rightarrow F_x$  is a linear map. In particular,  $g \mathcal{F}_t(t\Box^{X_i})(g^{-1}x, x)$  is well-defined on  $\tilde{U}_i$ . By (5.20), for  $y \in \tilde{U}_i^g$ , we have

$$(5.21) \quad \int_{Y \in N_{\tilde{U}_i^g/\tilde{U}_i, y}, |Y| < \epsilon_0} \tilde{\phi}_i(y, Y) \text{Tr}_s \left[ g \tilde{\mathcal{F}}_t(t\Box^Z)_{U_i, U_i}(g^{-1}(y, Y), (y, Y)) \right] k_i(y, Y) dY \\ = \int_{Y \in N_{\tilde{U}_i^g/\tilde{U}_i, y}, |Y| < \epsilon_0} \tilde{\phi}_i(y, Y) \text{Tr}_s \left[ g \mathcal{F}_t(t\Box^{X_i})(g^{-1}(y, Y), (y, Y)) \right] k_i(y, Y) dY \\ = t^{\frac{1}{2} \dim N_{\tilde{U}_i^g/\tilde{U}_i}} \int_{Y \in N_{\tilde{U}_i^g/\tilde{U}_i, y}, \sqrt{t}|Y| < \epsilon_0} \phi_i(y, \sqrt{t}Y) \\ \text{Tr}_s \left[ g \mathcal{F}_t(t\Box^{X_i})(g^{-1}(y, \sqrt{t}Y), (y, \sqrt{t}Y)) \right] k_i(y, \sqrt{t}Y) dY.$$

Let  $\left[ e \left( T\tilde{U}^g, \nabla^{T\tilde{U}^g} \right) \right]^{\max}$  be the function defined on  $\tilde{U}^g$  such that

$$(5.22) \quad e \left( T\tilde{U}^g, \nabla^{T\tilde{U}^g} \right) = \left[ e \left( T\tilde{U}^g, \nabla^{T\tilde{U}^g} \right) \right]^{\max} dv_{\tilde{U}^g}.$$

**Theorem 5.3.** *There is  $c > 0$  and  $C > 0$  such that for any  $i \in I$ ,  $g \in G_{U_i}$  and  $(y, \sqrt{t}Y) \in N_{\tilde{U}_i^g/\tilde{U}_i, \epsilon_0}$ , we have*

$$(5.23) \quad t^{\frac{1}{2} \dim N_{\tilde{U}_i^g/\tilde{U}_i}} \left| \tilde{\phi}_i \left( y, \sqrt{t}Y \right) \operatorname{Tr}_s \left[ g \mathcal{F}_t \left( t \square^{X_i} \right) \left( g^{-1}(y, \sqrt{t}Y), (y, \sqrt{t}Y) \right) \right] k_i \left( y, \sqrt{t}Y \right) \right| \leq C \exp(-c|Y|^2).$$

As  $t \rightarrow 0$ , we have

$$(5.24) \quad t^{\frac{1}{2} \dim N_{\tilde{U}_i^g/\tilde{U}_i}} \int_{\sqrt{t}Y \in N_{\tilde{U}_i^g/\tilde{U}_i, y, \epsilon_0}} \left\{ \tilde{\phi}_i(y, \sqrt{t}Y) \operatorname{Tr}_s \left[ g \mathcal{F}_t \left( t \square^{X_i} \right) \left( g^{-1}(y, \sqrt{t}Y), (y, \sqrt{t}Y) \right) \right] k_i \left( y, \sqrt{t}Y \right) dY \right\} \rightarrow \tilde{\phi}_i(y, 0) \operatorname{Tr}[\rho_{U_i}(g)] \left[ e \left( T\tilde{U}^g, \nabla^{T\tilde{U}^g} \right) \right]^{\max}.$$

*Proof.* Theorem 5.3 is a consequence of [BG01, Theorems 13.13 and 13.15].  $\square$

*The end of the proof of Theorem 4.4.* By (5.11), (5.14), (5.18), (5.21)-(5.24), and the dominated convergence Theorem, we get

$$(5.25) \quad \lim_{t \rightarrow 0} \operatorname{Tr}_s \left[ \exp \left( -t \square^Z \right) \right] = \sum_{i \in I} \frac{1}{|G_{U_i}|} \sum_{g \in G_{U_i}} \operatorname{Tr}[\rho_{U_i}(g)] \int_{\tilde{U}_i^g} \tilde{\phi}_i(y, 0) e \left( T\tilde{U}_i^g, \nabla^{T\tilde{U}_i^g} \right).$$

As  $\tilde{\phi}$  is  $G_{U_i}$ -invariant, the integrals on the right-hand side of (5.25) depends only on the conjugation class of  $G_{U_i}$ . Thus,

$$(5.26) \quad \sum_{i \in I} \frac{1}{|G_{U_i}|} \sum_{g \in G_{U_i}} \operatorname{Tr}[\rho_{U_i}(g)] \int_{\tilde{U}_i^g} \tilde{\phi}_i(y, 0) e \left( T\tilde{U}^g, \nabla^{T\tilde{U}^g} \right) = \sum_{i \in I} \sum_{[g] \in [G_{U_i}]} \frac{\operatorname{Tr}[\rho_{U_i}(g)]}{|Z_{G_{U_i}}(g)|} \int_{\tilde{U}_i^g} \tilde{\phi}_i(y, 0) e \left( T\tilde{U}^g, \nabla^{T\tilde{U}^g} \right) = \sum_{i=1}^l \rho_i \frac{\chi_{\text{orb}}(Z_i)}{m_i}.$$

By (5.25) and (5.26), we get (4.31). The proof of Theorem 4.4 is completed.  $\square$

For  $0 \leq q \leq m$ , denote by  $\square_q^Z$  the restriction of  $\square^Z$  to  $\Omega^q(Z, F)$ . For the sake of completeness, we prove the following corollary, which is also a consequence of [Ma05, Proposition 2.1].

**Corollary 5.4.** *There exist  $\{a_j \in \mathbf{R}\}_{j \in \mathbf{N}}$  such that for  $k \in \mathbf{N}$ , there is  $C_k > 0$  such that for  $t \in (0, 1]$ , we have*

$$(5.27) \quad \left| \operatorname{Tr} \left[ \exp \left( -t \square_q^Z \right) \right] - \frac{1}{t^{m/2}} \sum_{j=0}^k a_j t^{j/2} \right| \leq C_k t^{(k+1)/2}.$$

*Proof.* By (5.10) and (5.14), it is enough to show that

$$(5.28) \quad \int_{\tilde{U}_i} \tilde{\phi}_i(x) \operatorname{Tr} \left[ g \mathcal{F}_t \left( t \square_q^{X_i} \right) \left( g^{-1}x, x \right) \right] dv_{\tilde{U}_i}$$

satisfies (5.27). This is a consequence of short time asymptotic expansions of the heat kernel on compact manifolds [BeGeVe04, Theorem 2.30].  $\square$

**5.2. Proof of Theorem 4.11.** Following [BG01, P. 68], we introduce a new Grassmann variable  $\mathbf{z}$  which is anticommutative with  $ds$ . For two operators  $P, Q$  of trace class, set

$$(5.29) \quad \mathrm{Tr}^{\mathbf{z}}[P + \mathbf{z}Q] = \mathrm{Tr}[Q].$$

By (1.21), (4.53), (5.8) and (5.29), we have

$$(5.30) \quad u_t = \mathrm{Tr}_s^{\mathbf{z}} [\exp(-A_t^2 + \mathbf{z}B_t)] = \mathrm{Tr}_s^{\mathbf{z}} [\mathcal{F}_t(A_t^2 - \mathbf{z}B_t)] + \mathrm{Tr}_s^{\mathbf{z}} [\mathcal{G}_t(A_t^2 - \mathbf{z}B_t)].$$

We follow the same strategy used in the proof of Theorem 4.4. As in (5.10), proceeding as in [BG01, Theorem 13.6], when  $t \rightarrow 0$ , we have

$$(5.31) \quad \mathrm{Tr}_s^{\mathbf{z}} [\mathcal{G}_t(A_t^2 - \mathbf{z}B_t)] = \mathcal{O}(e^{-c/t}).$$

Moreover, the principal symbol of the lifting of  $A_t^2 - \mathbf{z}B_t$  on  $\tilde{U}_i$  is scalar, and is equal to  $t|\xi|^2/4$  for  $\xi \in T^*\tilde{U}_i$ . Take  $\alpha_0 < \min\{\delta_0, \epsilon_0\}$ . As in the case of  $\widetilde{\mathcal{F}}_t(t\Box^Z)$ , for  $x \in \tilde{U}$  and  $x \in \mathrm{Supp}(\tilde{\phi}_i)$ , the support of  $\widetilde{\mathcal{F}}_t(A_t^2 - \mathbf{z}B_t)_U(x, \cdot)$  is  $B_x^{\tilde{U}_i}(\alpha_0)$  and its value depends only on the restriction of  $A_t^2 - \mathbf{z}B_t$  on  $U_i$ . Also,

$$(5.32) \quad \mathrm{Tr}_s^{\mathbf{z}} [\mathcal{F}_t(A_t^2 - \mathbf{z}B_t)] = \sum_{i \in I} \frac{1}{|G_{U_i}|} \sum_{g \in G_{U_i}} \int_{y \in \tilde{U}_i^g} \left\{ \int_{Y \in N_{\tilde{U}^g/\tilde{U}_i, y}, |Y| < \epsilon_0} \tilde{\phi}_i(y, Y) \right. \\ \left. \mathrm{Tr}_s^{\mathbf{z}} \left[ g \widetilde{\mathcal{F}}_t(A_t^2 - \mathbf{z}B_t)_{U_i, U_i}(g^{-1}(y, Y), (y, Y)) \right] k_i(y, Y) dY \right\} dv_{\tilde{U}_i^g}$$

As in (5.19), we can replace  $A_t^2 - \mathbf{z}B_t$  by the corresponding operator on manifolds. Recall that  $h_g(\nabla^{\pi^* \tilde{F}_{U_i}}, g^{\pi^* \tilde{F}_{U_i}}) \in \Omega^{\mathrm{odd}}(\tilde{U}_i^g)$  is defined in (3.19). Proceeding as [BG01, Theorem 3.24], as  $t \rightarrow 0$ , we have

$$(5.33) \quad \int_{y \in \tilde{U}_i^g} \left\{ \int_{Y \in N_{\tilde{U}^g/\tilde{U}_i, y}, |Y| < \epsilon_0} \tilde{\phi}_i(y, Y) \mathrm{Tr}_s^{\mathbf{z}} \left[ g \widetilde{\mathcal{F}}_t(A_t^2 - \mathbf{z}B_t)_{U_i, U_i}(g^{-1}(y, Y), (y, Y)) \right] \right. \\ \left. k_i(y, Y) dY \right\} dv_{\tilde{U}_i^g} = \int_{\tilde{U}_i^g} \tilde{\phi}_i(y, 0) e\left(\pi^*(T\tilde{U}_i^g), \nabla^{\pi^*(T\tilde{U}_i^g)}\right) h_g\left(\nabla^{\pi^* \tilde{F}_{U_i}}, g^{\pi^* \tilde{F}_{U_i}}\right) + \mathcal{O}(\sqrt{t}).$$

Proceeding now as in (5.26), by (5.31)-(5.33), we get (4.62). The proof of Theorem 4.11 is completed.  $\square$

## 6. ANALYTIC TORSION ON COMPACT LOCALLY SYMMETRIC SPACE

Let  $G$  be a connected real reductive group with maximal compact subgroup  $K \subset G$ , and let  $\Gamma \subset G$  be a discrete cocompact subgroup of  $G$ . The corresponding locally symmetric space  $Z = \Gamma \backslash G/K$  is a compact orientable orbifold. The purpose of this section is to show Theorem 0.4 which claims an equality between the analytic torsion of an acyclic unitarily flat vector bundle  $F$  on  $Z$  and the zero value of the dynamical zeta function associated to the holonomy of  $F$ .

This section is organized as follows. In subsections 6.1 and 6.2, we recall some facts on reductive groups and the associated symmetric spaces.

In subsections 6.3 and 6.4, we recall the definition of semisimple elements and the semisimple orbital integrals.

In subsection 6.5, we introduce the discrete cocompact subgroup  $\Gamma$  and the associated locally symmetric spaces. We recall the Selberg trace formula.

In subsection 6.6, we introduce a Ruelle type dynamical zeta function associated to the holonomy of a unitarily flat vector bundle on  $Z$ . We restate Theorem 0.4. When the fundamental rank  $\delta(G) \in \mathbb{N}$  of  $G$  does not equal to 1, we show Theorem 0.4.

Subsections 6.7-6.11 are devoted to the case when  $\delta(G) = 1$ . In subsection 6.7, we recall some notation and results proved in [Sh16b, Sections 7.1 and 7.2].

In subsection 6.8, we introduce a class of representations of  $K$ . In subsection 6.9, using Bismut's formula [B11, Theorem 6.1.1], we evaluate the orbital integrals for the heat operators of the Casimir associated to the  $K$ -representations constructed in subsection 6.8.

In subsection 6.10, we introduce the Selberg type zeta functions, which are shown to be meromorphic on  $\mathbb{C}$  and satisfy certain functional equations.

In subsection 6.11, we show that the dynamical zeta function equals an alternating product of certain Selberg type zeta functions. We show Theorem 0.4.

**6.1. Reductive groups.** Let  $G$  be a linear connected real reductive group [Kn86, p. 3], let  $\theta \in \text{Aut}(G)$  be the Cartan involution. That means  $G$  is a closed connected group of real matrices that is stable under transpose, and  $\theta$  is the composition of transpose and inverse of matrices. Let  $K \subset G$  be the subgroup of  $G$  fixed by  $\theta$ , so that  $K$  is a maximal compact subgroup of  $G$ .

Let  $\mathfrak{g}$  and  $\mathfrak{k}$  be respectively the Lie algebras of  $G$  and  $K$ . The Cartan involution  $\theta$  acts naturally as Lie algebra automorphism of  $\mathfrak{g}$ . Then  $\mathfrak{k}$  is the eigenspace of  $\theta$  associated with the eigenvalue 1. Let  $\mathfrak{p} \subset \mathfrak{g}$  be the eigenspace with the eigenvalue  $-1$ , so that

$$(6.1) \quad \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}.$$

By [Kn86, Proposition 1.2], we have the diffeomorphism

$$(6.2) \quad (Y, k) \in \mathfrak{p} \times K \rightarrow e^Y k \in G.$$

Set

$$(6.3) \quad m = \dim \mathfrak{p}, \quad n = \dim \mathfrak{k}.$$

Let  $B$  be a real-valued nondegenerate bilinear symmetric form on  $\mathfrak{g}$  which is invariant under the adjoint action of  $G$ , and also under  $\theta$ . Then (6.1) is an orthogonal splitting with respect to  $B$ . We assume  $B$  to be positive on  $\mathfrak{p}$ , and negative on  $\mathfrak{k}$ . The form  $\langle \cdot, \cdot \rangle = -B(\cdot, \theta \cdot)$  defines an  $\text{Ad}(K)$ -invariant scalar product on  $\mathfrak{g}$  such that the splitting (6.1) is still orthogonal. We denote by  $|\cdot|$  the corresponding norm.

Let  $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$  be the complexification of  $\mathfrak{g}$  and let  $\mathfrak{u} = \sqrt{-1}\mathfrak{p} \oplus \mathfrak{k}$  be the compact form of  $\mathfrak{g}$ . Let  $G_{\mathbb{C}}$  and  $U$  be the connected group of complex matrices associated to the Lie algebras  $\mathfrak{g}_{\mathbb{C}}$  and  $\mathfrak{u}$ . By [Kn86, Propositions 5.3 and 5.6], if  $G$  has a compact center,  $G_{\mathbb{C}}$  is a linear connected complex reductive group with maximal compact subgroup  $U$ .

Let  $\mathcal{U}(\mathfrak{g})$  be the enveloping algebra of  $\mathfrak{g}$ . We identify  $\mathcal{U}(\mathfrak{g})$  with the algebra of left-invariant differential operators on  $G$ . Let  $C^{\mathfrak{g}} \in \mathcal{U}(\mathfrak{g})$  be the Casimir element. If  $e_1, \dots, e_m$  is an orthonormal basis of  $\mathfrak{p}$  and if  $e_{m+1}, \dots, e_{m+n}$  is an orthonormal basis of  $\mathfrak{k}$ , then

$$(6.4) \quad C^{\mathfrak{g}} = - \sum_{i=1}^m e_i^2 + \sum_{i=m+1}^{m+n} e_i^2.$$

Classically,  $C^{\mathfrak{g}}$  is in the center of  $\mathcal{U}(\mathfrak{g})$ .

We define  $C^{\mathfrak{k}}$  similarly. Let  $\tau$  be a finite dimensional representation of  $K$  on  $V$ . We denote by  $C^{\mathfrak{k},V}$  or  $C^{\mathfrak{k},\tau} \in \text{End}(V)$  the corresponding Casimir operator acting on  $V$ , so that

$$(6.5) \quad C^{\mathfrak{k},V} = C^{\mathfrak{k},\tau} = \sum_{i=m+1}^{m+n} \tau(e_i)^2.$$

Let  $\delta(G) \in \mathbb{N}$  be the fundamental rank of  $G$ , that is the difference between the complex ranks of  $G$  and  $K$ . If  $T \subset K$  is a maximal torus of  $K$  with Lie algebra of  $\mathfrak{t} \subset \mathfrak{k}$ , set

$$(6.6) \quad \mathfrak{b} = \{Y \in \mathfrak{p} : [Y, \mathfrak{t}] = 0\}.$$

Put

$$(6.7) \quad \mathfrak{h} = \mathfrak{b} \oplus \mathfrak{t}, \quad H = \exp(\mathfrak{b})T.$$

By [Kn86, Theorem 5.22],  $\mathfrak{h} \subset \mathfrak{g}$  (resp.  $H \subset G$ ) is a  $\theta$ -invariant Cartan subalgebra (resp. subgroup). Therefore,

$$(6.8) \quad \delta(G) = \dim \mathfrak{b}.$$

Moreover, up to conjugation,  $\mathfrak{h} \subset \mathfrak{g}$  (resp.  $H \subset G$ ) is the unique Cartan subalgebra (resp. subgroup) with minimal noncompact dimension.

**6.2. Symmetric space.** Let  $\omega^{\mathfrak{g}}$  be the canonical left-invariant 1-form on  $G$  with values in  $\mathfrak{g}$ , and let  $\omega^{\mathfrak{p}}, \omega^{\mathfrak{k}}$  be its components in  $\mathfrak{p}, \mathfrak{k}$ , so that

$$(6.9) \quad \omega^{\mathfrak{g}} = \omega^{\mathfrak{p}} + \omega^{\mathfrak{k}}.$$

Set  $X = G/K$ . Then  $p : G \rightarrow X = G/K$  is a  $K$ -principle bundle equipped with the connection form  $\omega^{\mathfrak{k}}$ .

Let  $\tau$  be a finite dimensional orthogonal representation of  $K$  on the real Euclidean space  $E_{\tau}$ . Let  $\mathcal{E}_{\tau}$  be the associated Euclidean vector bundle with total space  $G \times_K E_{\tau}$ . It is equipped a Euclidean connection  $\nabla^{\mathcal{E}_{\tau}}$  induced by  $\omega^{\mathfrak{k}}$ . We identify  $C^{\infty}(X, \mathcal{E}_{\tau})$  with the  $K$ -invariant subspace  $C^{\infty}(G, E_{\tau})^K$  of smooth  $E_{\tau}$ -valued functions on  $G$ . Let  $C^{\mathfrak{g},X,\tau}$  be the Casimir element of  $G$  acting on  $C^{\infty}(X, \mathcal{E}_{\tau})$ .

Observe that  $K$  acts isometrically on  $\mathfrak{p}$  by adjoint action. Using the above construction, the total space of the tangent bundle  $TX$  is given by

$$(6.10) \quad G \times_K \mathfrak{p}.$$

It is equipped with a Euclidean metric  $g^{TX}$  and a Euclidean connection  $\nabla^{TX}$ , which coincides with the Levi-Civita connection of the Riemannian manifold  $(X, g^{TX})$ . Classically,  $(X, g^{TX})$  has non positive sectional curvature.

If  $E_{\tau} = \Lambda^{\bullet}(\mathfrak{p}^*)$  is equipped with the  $K$ -action induced by the adjoint action, then  $C^{\infty}(X, \mathcal{E}_{\tau}) = \Omega^{\bullet}(X)$ . In this case, we write  $C^{\mathfrak{g},X} = C^{\mathfrak{g},X,\tau}$ . By [B11, Proposition 7.8.1],  $C^{\mathfrak{g},X}$  coincides with the Hodge Laplacian acting on  $\Omega^{\bullet}(X)$ .

Let  $dv_X$  be the Riemannian volume of  $(X, g^{TX})$ . Define  $[e(TX, \nabla^{TX})]_{\max}$  as in (5.22). Since both  $dv_X$  and  $e(TX, \nabla^{TX})$  are  $G$ -invariant,  $[e(TX, \nabla^{TX})]_{\max} \in \mathbb{R}$  is a constant. Note that  $\delta(G)$  and  $\dim X$  have the same parity. By [Sh16b, Proposition 3.1], if  $\delta(G) \neq 0$ , then

$$(6.11) \quad [e(TX, \nabla^{TX})]_{\max} = 0.$$

If  $\delta(G) = 0$ ,  $G$  has a compact center. Then  $U$  is a compact group with maximal torus  $T$ . Denote by  $W(T, U)$  (resp.  $W(T, K)$ ) the Weyl group of  $U$  (resp.  $K$ ) with respect to

$T$ , and by  $\text{vol}(U/K)$  the volume of  $U/K$  induced by  $-B$ . Then, [Sh16b, Proposition 3.1] asserts

$$(6.12) \quad [e(TX, \nabla^{TX})]^{\max} = (-1)^{m/2} \frac{|W(T, U)|/|W(T, K)|}{\text{vol}(U/K)}.$$

**6.3. Semisimple elements.** If  $\gamma \in G$ , we denote by  $Z(\gamma) \subset G$  the centralizer of  $\gamma$  in  $G$ , and by  $\mathfrak{z}(\gamma) \subset \mathfrak{g}$  its Lie algebra. If  $a \in \mathfrak{g}$ , let  $Z(a) \subset G$  be the stabilizer of  $a$  in  $G$ , and let  $\mathfrak{z}(a) \subset \mathfrak{g}$  be its Lie algebra.

Following [B11, Section 3.1],  $\gamma \in G$  is said to be semisimple if and only if there is  $g_\gamma \in G$ , such that  $\gamma = g_\gamma e^a k^{-1} g_\gamma^{-1}$  with

$$(6.13) \quad a \in \mathfrak{p}, \quad k \in K, \quad \text{Ad}(k)a = a.$$

Set

$$(6.14) \quad a_\gamma = \text{Ad}(g_\gamma)a, \quad k_\gamma = g_\gamma k g_\gamma^{-1}.$$

Therefore,  $\gamma = e^{a_\gamma} k_\gamma^{-1}$ . Moreover, this decomposition does not depend on the choice of  $g_\gamma$ . By [B11, (3.3.3)], we have

$$(6.15) \quad Z(\gamma) = Z(a_\gamma) \cap Z(k_\gamma), \quad \mathfrak{z}(\gamma) = \mathfrak{z}(a_\gamma) \cap \mathfrak{z}(k_\gamma).$$

By [Kn02, Proposition 7.25],  $Z(\gamma)$  is reductive. The corresponding Cartan evolution and bilinear form are given by

$$(6.16) \quad \theta_{g_\gamma} = g_\gamma \theta g_\gamma^{-1}, \quad B_{g_\gamma}(\cdot, \cdot) = B(\text{Ad}(g_\gamma^{-1})\cdot, \text{Ad}(g_\gamma^{-1})\cdot).$$

Let  $K(\gamma) \subset Z(\gamma)$  be the fixed point of  $\theta_{g_\gamma}$ , so  $K(\gamma)$  is a maximal compact subgroup  $Z(\gamma)$ . Let  $\mathfrak{k}(\gamma) \subset \mathfrak{z}(\gamma)$  be the Lie algebra of  $K(\gamma)$ . Let

$$(6.17) \quad \mathfrak{z}(\gamma) = \mathfrak{p}(\gamma) \oplus \mathfrak{k}(\gamma)$$

be the Cartan decomposition of  $\mathfrak{z}(\gamma)$ . Let

$$(6.18) \quad X(\gamma) = Z(\gamma)/K(\gamma)$$

be the associated symmetric space.

Let  $Z^0(\gamma)$  be the connected component of the identity in  $Z(\gamma)$ . Similarly,  $Z^0(\gamma)$  is reductive with maximal compact subgroup  $Z^0(\gamma) \cap K(\gamma)$ . Also,  $Z^0(\gamma) \cap K(\gamma)$  coincides with  $K^0(\gamma)$ , the connected component of the identity in  $K(\gamma)$ . Clearly, we have

$$(6.19) \quad X(\gamma) = Z^0(\gamma)/K^0(\gamma).$$

The semisimple element  $\gamma$  is called elliptic if  $a_\gamma = 0$ . Assume now  $\gamma$  is semisimple and nonelliptic. Then  $a_\gamma \neq 0$ . Let  $\mathfrak{z}^{a,\perp}(\gamma)$  (resp.  $\mathfrak{p}^{a,\perp}(\gamma)$ ) be the orthogonal spaces to  $a_\gamma$  in  $\mathfrak{z}(\gamma)$  (resp.  $\mathfrak{p}(\gamma)$ ) with respect to  $B_{g_\gamma}$ . Thus,

$$(6.20) \quad \mathfrak{z}^{a,\perp}(\gamma) = \mathfrak{p}^{a,\perp}(\gamma) \oplus \mathfrak{k}(\gamma).$$

Moreover,  $\mathfrak{z}^{a,\perp}(\gamma)$  is a Lie algebra. Let  $Z^{a,\perp,0}(\gamma)$  be the connected subgroup of  $Z^0(\gamma)$  that is associated with the Lie algebra  $\mathfrak{z}^{a,\perp}(\gamma)$ . By [B11, (3.3.11)],  $Z^{a,\perp,0}(\gamma)$  is reductive with maximal compact subgroup  $K^0(\gamma)$  with Cartan decomposition (6.20), and

$$(6.21) \quad Z^0(\gamma) = \mathbf{R} \times Z^{a,\perp,0}(\gamma),$$

so that  $e^{ta_\gamma} \in Z^0(\gamma)$  maps into  $t|a| \in \mathbf{R}$ . Set

$$(6.22) \quad X^{a,\perp}(\gamma) = Z^{a,\perp,0}(\gamma)/K^0(\gamma).$$



By (6.19), (6.21) and (6.22), we have

$$(6.23) \quad X(\gamma) = \mathbf{R} \times X^{a,\perp}(\gamma),$$

so that the action  $e^{ta_\gamma}$  on  $X(\gamma)$  is just the translation by  $t|a|$  on  $\mathbf{R}$ .

**6.4. Semisimple orbital integral.** Recall that  $\tau$  is a finite dimensional orthogonal representation of  $K$  on the real Euclidean space  $E_\tau$ . Let  $p_t^{X,\tau}(x, x')$  be the smooth kernel of  $\exp(-tC^{\mathfrak{g},X,\tau}/2)$  with respect to the Riemannian volume  $dv_X$ . We define a  $K \times K$ -invariant function in  $C^\infty(G, \text{End}(E_\tau))$  by

$$(6.24) \quad p_t^{X,\tau}(g) = p_t^{X,\tau}(p1, pg).$$

Let  $dv_G$  be the left-invariant Riemannian volume on  $G$  induced by  $-B(\cdot, \theta \cdot)$ . For a semisimple element  $\gamma \in G$ , denote by  $dv_{Z^0(\gamma)}$  the left-invariant Riemannian volume on  $Z^0(\gamma)$  induced by  $-B_{g_\gamma}(\cdot, \theta_{g_\gamma} \cdot)$ . Clearly, the choice of  $g_\gamma$  is irrelevant. Let  $dv_{Z^0(\gamma) \backslash G}$  be the Riemannian volume on  $Z^0(\gamma) \backslash G$  such that  $dv_G = dv_{Z^0(\gamma)} dv_{Z^0(\gamma) \backslash G}$ . By [B11, Definition 4.2.2, Proposition 4.4.2], the orbital integral

$$(6.25) \quad \text{Tr}^{[\gamma]} [\exp(-tC^{\mathfrak{g},X,\tau}/2)] = \frac{1}{\text{vol}(K^0(\gamma) \backslash K)} \int_{Z^0(\gamma) \backslash G} \text{Tr}^{E_\tau} [p_t^{X,\tau}(g)] dv_{Z^0(\gamma) \backslash G}$$

is well-defined.

*Remark 6.1.* In [B11, Definition 4.2.2], the volume forms are normalized such that  $\text{vol}(K_0(\gamma) \backslash K) = 1$ .

*Remark 6.2.* As the notation  $\text{Tr}^{[\gamma]}$  indicates, the orbital integral only depends on the conjugacy class of  $\gamma$  in  $G$ . However, the notation  $[\gamma]$  will be used later for the conjugacy class of a discrete group  $\Gamma$ . Here, we consider  $\text{Tr}^{[\gamma]}$  as an abstract symbol.

We will also consider the case where  $E_\tau = E_\tau^+ \oplus E_\tau^-$  is a  $\mathbf{Z}_2$ -graded representation of  $K$ . In this case, We will use the notation  $\text{Tr}_s^{[\gamma]} [\exp(-tC^{\mathfrak{g},X,\tau}/2)]$  when the trace on the right-hand side of (6.25) is replaced by the supertrace on  $E_\tau$ .

In [B11, Theorem 6.1.1], for any semisimple element  $\gamma \in G$ , Bismut gave an explicit formula for  $\text{Tr}^{[\gamma]} [\exp(-tC^{\mathfrak{g},X,\tau}/2)]$ . For the later use, let us recall the formula when  $\gamma$  is elliptic.

Assume now  $\gamma \in K$ . By (6.14), we can take  $g_\gamma = 1$ . Then  $\mathfrak{p}(\gamma) \subset \mathfrak{p}$ ,  $\mathfrak{k}(\gamma) \subset \mathfrak{k}$ . Let  $\mathfrak{p}^\perp(\gamma) \subset \mathfrak{p}$ ,  $\mathfrak{k}^\perp(\gamma) \subset \mathfrak{k}$  be the orthogonal space of  $\mathfrak{p}(\gamma)$ ,  $\mathfrak{k}(\gamma)$ . Take  $\mathfrak{z}^\perp(\gamma) = \mathfrak{p}^\perp(\gamma) \oplus \mathfrak{k}^\perp(\gamma)$ . Recall that  $\widehat{A}$  is the holomorphic function defined in (1.15). Following [B11, Theorem 5.5.1], for  $Y \in \mathfrak{k}(\gamma)$ , put

$$(6.26) \quad J_\gamma(Y) = \frac{\widehat{A}(i \text{ad}(Y)|_{\mathfrak{p}(\gamma)})}{\widehat{A}(i \text{ad}(Y)|_{\mathfrak{k}(\gamma)})} \left[ \frac{1}{\det(1 - \text{Ad}(\gamma))|_{\mathfrak{z}^\perp(\gamma)}} \frac{\det(1 - \exp(-i \text{ad}(Y)) \text{Ad}(\gamma))|_{\mathfrak{k}^\perp(\gamma)}}{\det(1 - \exp(-i \text{ad}(Y)) \text{Ad}(\gamma))|_{\mathfrak{p}^\perp(\gamma)}} \right]^{1/2}.$$

Note that by [B11, Section 5.5], the square root in (6.26) is well-defined, and its sign is chosen such that

$$(6.27) \quad J_\gamma(0) = \left( \det(1 - \exp(-i \text{ad}(Y)) \text{Ad}(\gamma))|_{\mathfrak{p}^\perp(\gamma)} \right)^{-1}.$$

Moreover,  $J_\gamma$  is an  $\text{Ad}(K^0(\gamma))$ -invariant analytic function on  $\mathfrak{k}(\gamma)$  such that there exist  $c_\gamma > 0, C_\gamma > 0$ , for  $Y \in \mathfrak{k}(\gamma)$ ,

$$(6.28) \quad |J_\gamma(Y)| \leq C_\gamma \exp(c_\gamma |Y|).$$

Denote by  $dY$  be the Lebesgue measure on  $\mathfrak{k}(\gamma)$  induced by  $-B$ . Recall that  $C^{\mathfrak{k}, \mathfrak{p}}, C^{\mathfrak{k}, \mathfrak{k}}$  are defined in (6.5). By [B11, Theorem 6.1.1], for  $t > 0$ , we have

$$(6.29) \quad \text{Tr}^{[\gamma]} [\exp(-tC^{\mathfrak{g}, X, \tau}/2)] = \frac{1}{(2\pi t)^{\dim \mathfrak{z}(\gamma)/2}} \exp\left(\frac{t}{16} \text{Tr}^{\mathfrak{p}}[C^{\mathfrak{k}, \mathfrak{p}}] + \frac{t}{48} \text{Tr}^{\mathfrak{k}}[C^{\mathfrak{k}, \mathfrak{k}}]\right) \\ \int_{Y \in \mathfrak{k}(\gamma)} J_\gamma(Y) \text{Tr}^{E_\tau}[\tau(\gamma) \exp(-i\tau(Y))] \exp(-|Y|^2/2t) dY.$$

**6.5. Locally symmetric spaces.** Let  $\Gamma \subset G$  be a discrete cocompact subgroup of  $G$ . Then the elements of  $\Gamma$  are semisimple. Let  $\Gamma_e \subset \Gamma$  be the subset of elliptic elements in  $\Gamma$ . Set  $\Gamma_+ = \Gamma - \Gamma_e$ .

The group  $\Gamma$  acts properly discontinuously and isometrically on the left on  $X$ . Take  $Z = \Gamma \backslash X$  to be the corresponding locally symmetric space. By Proposition 2.12,  $Z$  is a compact orbifold with orbifold fundamental group

$$(6.30) \quad \Gamma' = \Gamma / \ker(\Gamma \rightarrow \text{Diffeo}(X)).$$

Moreover,  $X$  is the universal covering orbifold of  $Z$ . The Riemannian metric  $g^{TX}$  on  $X$  induces a Riemannian metric  $g^{TZ}$  on  $Z$ . Clearly,  $(Z, g^{TZ})$  has nonpositive curvature.

Let  $F$  be a (possibly non proper) flat vector bundle on  $Z$  with holonomy  $\rho' : \Gamma' \rightarrow \text{GL}_r(\mathbb{C})$  such that

$$(6.31) \quad C^\infty(Z, F) = C^\infty(X, \mathbb{C}^r)^{\Gamma'}.$$

Take  $\rho$  to be the composition of the projection  $\Gamma \rightarrow \Gamma'$  and  $\rho'$ . Then

$$(6.32) \quad C^\infty(Z, F) = C^\infty(X, \mathbb{C}^r)^\Gamma.$$

By abuse of notation, we still call also  $\rho : \Gamma \rightarrow \text{GL}_r(\mathbb{C})$  the holonomy of  $F$ . In the rest of this section, we assume  $F$  is unitarily flat, or equivalently  $\rho$  is unitary. Let  $g^F$  be the canonical Hermitian metric on  $F$  induced by the canonical Hermitian metric on  $\mathbb{C}^r$  via (6.32). As  $g^{TZ}$  and  $g^F$  are fixed in the whole section, we denote by

$$(6.33) \quad T(F) = T(F, g^{TZ}, g^F).$$

The group  $\Gamma$  acts on the Euclidean vector bundles like  $\mathcal{E}_\tau$ , and preserves the corresponding connections  $\nabla^{\mathcal{E}_\tau}$ . The vector bundle  $\mathcal{E}_\tau$  descends to a (possibly non proper) orbifold vector bundle  $\mathcal{F}_\tau$  on  $Z$ . The total space of  $\mathcal{F}_\tau$  is given by  $\Gamma \backslash G \times_K E_\tau$ , and we have the identification of vector spaces

$$(6.34) \quad C^\infty(Z, \mathcal{F}_\tau) \simeq C^\infty(\Gamma \backslash G, E_\tau)^K.$$

By (6.32) and (6.34), we identify  $C^\infty(Z, \mathcal{F}_\tau \otimes_{\mathbb{R}} F)$  with the  $\Gamma$ -invariant subspace of  $C^\infty(X, \mathcal{E}_\tau \otimes_{\mathbb{R}} \mathbb{C}^r)$ . Let  $C^{\mathfrak{g}, Z, \tau, \rho}$  be the Casimir operator of  $G$  acting on  $C^\infty(Z, \mathcal{F}_\tau \otimes_{\mathbb{R}} F)$ . As we see in subsection 6.2, when  $E_\tau = \Lambda(\mathfrak{p}^*)$ ,

$$(6.35) \quad \Omega(Z, F) \simeq C^\infty(Z, \mathcal{F}_\tau \otimes_{\mathbb{R}} F),$$

and the Hodge Laplacian acting on  $\Omega(Z, F)$  is given by

$$(6.36) \quad \square^Z = C^{\mathfrak{g}, Z, \tau, \rho}.$$

For  $\gamma \in \Gamma$ , set  $\Gamma(\gamma) = Z(\gamma) \cap \Gamma$ . It is well-known (c.f. [Sh16b, Proposition 3.9]),  $\Gamma(\gamma)$  is cocompact in  $Z(\gamma)$ . Then  $\Gamma(\gamma) \backslash X(\gamma)$  is a compact locally symmetric orbifold. Clearly, it depends only on the conjugacy class of  $\gamma$  in  $\Gamma$ . Denote by  $\text{vol}(\Gamma(\gamma) \backslash X(\gamma))$  the Riemannian volume of  $\Gamma(\gamma) \backslash X(\gamma)$  induced by  $B_{g_\gamma}$ . Let  $\delta(\gamma) \subset K(\gamma)$  be the subgroup of  $K(\gamma)$  that acts on the right like the identity on  $\Gamma(\gamma) \backslash Z(\gamma)$ . Then  $\delta(\gamma)$  is a finite subgroup of  $\Gamma \cap K(\gamma)$ .

For  $\gamma \in \Gamma$ , denote by  $[\gamma] \in [\Gamma]$  its conjugacy class. Let  $[\Gamma]$  be the set of conjugation classes of  $\Gamma$ , and let  $[\Gamma_e] \subset [\Gamma]$  and  $[\Gamma_+] \subset [\Gamma]$  be respectively the subsets of  $[\Gamma]$  formed by the conjugation classes of elements in  $\Gamma_e$  and  $\Gamma_+$ . Clearly,  $[\Gamma_e]$  is a finite set.

**Theorem 6.3.** *There exist  $c > 0$  and  $C > 0$  such that*

$$(6.37) \quad \sum_{[\gamma] \in [\Gamma_+]} \frac{\text{vol}(\Gamma(\gamma) \backslash X(\gamma))}{|\delta(\gamma)|} \left| \text{Tr}^{[\gamma]} [\exp(-tC^{\mathfrak{g}, X, \tau}/2)] \right| \leq C \exp\left(-\frac{c}{t} + Ct\right).$$

The following identity holds:

$$(6.38) \quad \text{Tr} [\exp(-tC^{\mathfrak{g}, Z, \tau, \rho}/2)] = \sum_{[\gamma] \in [\Gamma]} \text{Tr}[\rho(\gamma)] \frac{\text{vol}(\Gamma(\gamma) \backslash X(\gamma))}{|\delta(\gamma)|} \text{Tr}^{[\gamma]} [\exp(-tC^{\mathfrak{g}, X, \tau}/2)].$$

*Proof.* The proof is identical to the proof of [Sh16b, Theorem 3.10].  $\square$

**6.6. Dynamical zeta functions.** The geodesic flow on the unit tangent bundle of  $Z$  is still well-defined. Proceeding as in the proof for the manifold case [DuKoV79, Proposition 5.15], the fixed points of the geodesic flow consists of a disjoint union of smooth connected compact suborbifold

$$(6.39) \quad \coprod_{[\gamma] \in [\Gamma_+]} B_{[\gamma]}.$$

Moreover,  $B_{[\gamma]}$  is diffeomorphic to  $\Gamma(\gamma) \backslash X(\gamma)$ . Also, all the elements in  $B_{[\gamma]}$  have the same length  $l_{[\gamma]} = |a_\gamma| > 0$ .

The geodesic flow induces a locally free  $\mathbb{S}^1$ -action on  $B_{[\gamma]}$ . Then  $\mathbb{S}^1 \backslash B_{[\gamma]}$  is an orbifold. Set

$$(6.40) \quad m_{[\gamma]} = |\delta(\gamma)| \left| \ker \left( \mathbb{S}^1 \rightarrow \text{Diffeo}(B_{[\gamma]}) \right) \right|.$$

**Proposition 6.4.** *For  $\gamma \in \Gamma_+$  such that  $\gamma = a_\gamma k_\gamma^{-1}$  as in (6.14), we have*

$$(6.41) \quad \frac{\chi_{\text{orb}}(\mathbb{S}^1 \backslash B_{[\gamma]})}{m_{[\gamma]}} = \frac{\text{vol}(\Gamma(\gamma) \backslash X(\gamma))}{|a_\gamma| |\delta(\gamma)|} \left[ e \left( TX^{a, \perp}(\gamma), \nabla^{TX^{a, \perp}(\gamma)} \right) \right]^{\max}$$

In particular, if  $\delta(G) \geq 2$ , then for all  $[\gamma] \in [\Gamma_+]$ , we have

$$(6.42) \quad \chi_{\text{orb}}(\mathbb{S}^1 \backslash B_{[\gamma]}) = 0.$$

Also, if  $\delta(G) = 1$  and if  $\gamma$  can not be conjugated into  $H$ , then (6.42) still holds.

*Proof.* The proof (6.41) is identical to the one given in [Sh16b, Proposition 4.1, Corollary 4.2].  $\square$

Recall that  $\rho : \Gamma \rightarrow U(r)$  is a unitaire representation of  $\Gamma$ .

**Definition 6.5.** The dynamical zeta function  $R_\rho$  is said to be well-defined if

- for  $\operatorname{Re}(\sigma) \gg 1$ , the sum

$$(6.43) \quad \Xi_\rho(\sigma) = \sum_{[\gamma] \in [\Gamma_+]} \operatorname{Tr}[\rho(\gamma)] \frac{\chi_{\text{orb}}(\mathbb{S}^1 \setminus B_{[\gamma]})}{m_{[\gamma]}} e^{-\sigma l_{[\gamma]}}$$

converges absolutely to a holomorphic function of  $\sigma$ .

- the function  $R_\rho(\sigma) = \exp(\Xi_\rho(\sigma))$  has a meromorphic extension to  $\sigma \in \mathbb{C}$ .

By (6.42), if  $\delta(G) \geq 2$ , the dynamical zeta function  $R_\rho$  is well-defined and

$$(6.44) \quad R_\rho \equiv 1.$$

We restate Theorem 0.4, which is the main result of this section.

**Theorem 6.6.** *If  $\dim Z$  is odd, then the dynamical zeta function  $R_\rho(\sigma)$  is a well-defined. There exist explicit constant  $C_\rho \in \mathbb{R}$  with  $C_\rho \neq 0$  and  $r_\rho \in \mathbb{Z}$  (c.f. (6.97)) such that as  $\sigma \rightarrow 0$ ,*

$$(6.45) \quad R_\rho(\sigma) = C_\rho T(F)^2 \sigma^{r_\rho} + \mathcal{O}(\sigma^{r_\rho+1}).$$

Moreover, if  $H^*(Z, F) = 0$ , we have

$$(6.46) \quad C_\rho = 1, \quad r_\rho = 0,$$

so that

$$(6.47) \quad R_\rho(0) = T(F)^2.$$

*Proof.* As  $\dim Z$  is odd,  $\delta(G)$  is an odd integral. If  $\delta(G) \geq 3$ , by [B11, Theorem 7.9.1], for any  $\gamma \in G$  semisimple, we have

$$(6.48) \quad \operatorname{Tr}_s^{[\gamma]} \left[ \left( N^{\Lambda^*(T^*X)} - \frac{m}{2} \right) \exp(-tC^{\mathfrak{g},X}/2) \right] = 0.$$

From (6.36), (6.38), (6.48), we can deduce that

$$(6.49) \quad T(F) = 1.$$

From (6.44) and (6.49), we get (6.47).

The proof for the case  $\delta(G) = 1$  will be given in subsections 6.8-6.11. □

*Remark 6.7.* By (6.47), we have the formal identity

$$(6.50) \quad \log T(F)^2 = \sum_{[\gamma] \in [\Gamma_+]} \operatorname{Tr}[\rho(\gamma)] \frac{\chi_{\text{orb}}(\mathbb{S}^1 \setminus B_{[\gamma]})}{m_{[\gamma]}}.$$

We note the similarity between (4.32) and (6.50).

*Remark 6.8.* The formal identity (6.50) can be deduced formally using the path integral argument and Bismut-Goette's  $V$ -invariant as in [Sh16b, Introduction 0.5]. We leave the details to readers.

**6.7. Reductive group with fundamental rank 1.** From now on, we assume that  $\delta(G) = 1$ . Let us introduce some notations following [Sh16b, Sections 5.1, 5.2]. For  $a \in \mathfrak{b}$  and  $a \neq 0$ , set

$$(6.51) \quad Z(\mathfrak{b}) = Z(e^a), \quad M = Z^{a, \perp, 0}(e^a), \quad K_M = K \cap M,$$

and

$$(6.52) \quad \mathfrak{z}(\mathfrak{b}) = \mathfrak{z}(e^a), \quad \mathfrak{m} = \mathfrak{z}^{a, \perp}(e^a), \quad \mathfrak{p}_m = \mathfrak{m} \cap \mathfrak{p}, \quad \mathfrak{k}_m = \mathfrak{m} \cap \mathfrak{k}.$$

Then  $M$  is a connected reductive Lie group with maximal compact subgroup  $K_M$  with Cartan decomposition  $\mathfrak{m} = \mathfrak{p}_m \oplus \mathfrak{k}_m$ .

Let  $\mathfrak{p}^\perp(\mathfrak{b}) \subset \mathfrak{p}$ ,  $\mathfrak{k}^\perp(\mathfrak{b}) \subset \mathfrak{k}$  be respectively the orthogonal spaces of  $\mathfrak{b} \oplus \mathfrak{p}_m$ ,  $\mathfrak{k}_m$ . Then,

$$(6.53) \quad \mathfrak{p} = \mathfrak{b} \oplus \mathfrak{p}_m \oplus \mathfrak{p}^\perp(\mathfrak{b}), \quad \mathfrak{k} = \mathfrak{k}_m \oplus \mathfrak{k}^\perp(\mathfrak{b}).$$

Set

$$(6.54) \quad \mathfrak{z}^\perp(\mathfrak{b}) = \mathfrak{p}^\perp(\mathfrak{b}) \oplus \mathfrak{k}^\perp(\mathfrak{b}).$$

By [Sh16b, Proposition 5.2], if  $G$  has a compact center, there exists  $\alpha \in \mathfrak{b}^*$  such that for any  $a \in \mathfrak{b}$ , the action of  $\text{ad}(a)$  on  $\mathfrak{z}^\perp(\mathfrak{b})$  has only two eigenvalues  $\pm \langle \alpha, a \rangle \in \mathbb{R}$ . We fix  $a_0 \in \mathfrak{b}$  such that

$$(6.55) \quad \langle \alpha, a_0 \rangle = 1.$$

Let  $\mathfrak{n} \subset \mathfrak{z}^\perp(\mathfrak{b})$  (resp.  $\bar{\mathfrak{n}}$ ) be the  $+1$  (resp.  $-1$ ) eigenspace of  $\text{ad}(a_0)$ , so that

$$(6.56) \quad \mathfrak{z}^\perp(\mathfrak{b}) = \mathfrak{n} \oplus \bar{\mathfrak{n}}.$$

Clearly,  $\bar{\mathfrak{n}} = \theta \mathfrak{n}$ , and  $M$  acts on  $\mathfrak{n}$  and  $\bar{\mathfrak{n}}$ . If  $G$  has a noncompact center, then  $\mathfrak{z}^\perp(\mathfrak{b}) = 0$ . We take  $\mathfrak{n} = 0$ ,  $\alpha = 0$ , and fix  $a_0 \in \mathfrak{b}^*$  once for all.

As explained in [Sh16b, Section 5.1],  $\dim \mathfrak{n}$  is even. Set

$$(6.57) \quad l = \frac{1}{2} \dim \mathfrak{n}.$$

Let  $\mathfrak{u}(\mathfrak{b}) \subset \mathfrak{u}$  and  $\mathfrak{u}_m \subset \mathfrak{u}$  be respectively the compact forms of  $\mathfrak{z}(\mathfrak{b})$  and of  $\mathfrak{m}$ . Then,

$$(6.58) \quad \mathfrak{u}(\mathfrak{b}) = \sqrt{-1}\mathfrak{b} \oplus \sqrt{-1}\mathfrak{p}_m \oplus \mathfrak{k}_m, \quad \mathfrak{u}_m = \sqrt{-1}\mathfrak{p}_m \oplus \mathfrak{k}_m.$$

Let  $\mathfrak{u}^\perp(\mathfrak{b}) \subset \mathfrak{u}$  be the orthogonal space of  $\mathfrak{u}(\mathfrak{b})$ . Then we have

$$(6.59) \quad \mathfrak{u} = \sqrt{-1}\mathfrak{b} \oplus \mathfrak{u}_m \oplus \mathfrak{u}^\perp(\mathfrak{b}).$$

Let  $U(\mathfrak{b}) \subset U$  and  $U_M \subset U$  be respectively the corresponding connected subgroups (possibly non closed) of complex matrices of groups associated to the Lie algebras  $\mathfrak{u}(\mathfrak{b})$  and  $\mathfrak{u}_m$ . Note that as  $\delta(M) = 0$ ,  $U_M$  is compact. The group  $U_M$  acts  $\mathfrak{u}_m$  and  $\mathfrak{u}^\perp(\mathfrak{b})$ , and acts trivially on  $\mathfrak{b}$ . As  $U(\mathfrak{b}) = \exp(\sqrt{-1}\mathfrak{b})U_M$ , we extends the actions of  $U_M$  on  $\mathfrak{b}$ ,  $\mathfrak{u}_m$  and  $\mathfrak{u}^\perp(\mathfrak{b})$  to  $U(\mathfrak{b})$  such that  $\exp(\sqrt{-1}\mathfrak{b})$  acts trivially.

Put

$$(6.60) \quad Y_{\mathfrak{b}} = U/U(\mathfrak{b}).$$

By [Sh16b, Propositions 5.2, 5.7], if  $G$  has a noncompact center,  $Y_{\mathfrak{b}}$  is a point, and if  $G$  has a compact center,  $Y_{\mathfrak{b}}$  is a Hermitian symmetric space of the compact type. Let  $\omega^{Y_{\mathfrak{b}}} \in \Omega^2(Y_{\mathfrak{b}})$  be the canonical Kähler form on  $Y_{\mathfrak{b}}$  induced by  $B$ . As in subsection 6.2,  $U \rightarrow Y_{\mathfrak{b}}$  is a  $U(\mathfrak{b})$ -principle bundle on  $Y_{\mathfrak{b}}$  with canonical connection. Let  $(TY_{\mathfrak{b}}, \nabla^{TY_{\mathfrak{b}}})$  and  $(N_{\mathfrak{b}}, \nabla^{N_{\mathfrak{b}}})$  the Hermitian vector bundle with Hermitian connection induced by the

representation of  $U(\mathfrak{b})$  on  $\mathfrak{u}_{\mathfrak{m}}$  and  $\mathfrak{u}_{\mathfrak{m}}^{\perp}$ . For a vector space  $E$ , denote by the trivial bundle  $\underline{E}$  on  $Y_{\mathfrak{b}}$ . By (6.59), we have an analogy of [B11, (2.2.1)],

$$(6.61) \quad \underline{\mathfrak{u}} = \underline{\mathbf{R}} \oplus N_{\mathfrak{b}} \oplus TY_{\mathfrak{b}}.$$

**6.8. Auxiliary virtual representations of  $K$ .** We follow [Sh16b, Section 5.3]. Denote by  $R(K_M, \mathbf{R})$ ,  $R(K, \mathbf{R})$  the real representation rings of  $K_M$  and  $K$ . As  $K_M$  and  $K$  have the same maximal torus  $T$ , the restriction  $R(K, \mathbf{R}) \rightarrow R(K_M, \mathbf{R})$  is injective.

Let  $\eta$  be a real finite dimensional representation of  $M$  on the vector space  $E_{\eta}$  satisfying the following assumptions.

**Assumption 6.9.** *We assume that:*

- (1) *The restriction  $\eta|_{K_M}$  to  $K_M$  can be lifted into  $R(K, \mathbf{R})$ ;*
- (2) *The action of the Lie algebra  $\mathfrak{u}_{\mathfrak{m}} \subset \mathfrak{m} \otimes_{\mathbf{R}} \mathbf{C}$  on  $E_{\eta} \otimes_{\mathbf{R}} \mathbf{C}$ , induced by complexification, can be lifted to an action of Lie group  $U_M$ ;*
- (3) *The Casimir element  $C^{\mathfrak{u}_{\mathfrak{m}}}$  of  $\mathfrak{u}_{\mathfrak{m}}$  acts on  $E_{\eta} \otimes_{\mathbf{R}} \mathbf{C}$  as a scalar  $C^{\mathfrak{u}_{\mathfrak{m}}, \eta} \in \mathbf{R}$ .*

By (2) of the Assumption 6.9,  $U_M$  on  $E_{\eta} \otimes_{\mathbf{R}} \mathbf{C}$ . We extend this action to  $U(\mathfrak{b})$  such that  $\exp(\sqrt{-1}\mathfrak{b})$  acts trivially. Denote by  $F_{\mathfrak{b}, \eta}$  the Hermitian vector bundle on  $Y_{\mathfrak{b}}$  with total space  $U \times_{U(\mathfrak{b})} (E_{\eta} \otimes_{\mathbf{R}} \mathbf{C})$  with Hermitian connection  $\nabla^{F_{\mathfrak{b}, \eta}}$ .

Note that  $K_M$  acts on  $\mathfrak{p}_{\mathfrak{m}}$  by adjoint action. By [Sh16b, Theorem 5.11], the induced action of  $K_M$  on  $\Lambda^*(\mathfrak{p}_{\mathfrak{m}}^*)$  has a unique lift in  $R(K, \mathbf{R})$ .

**Definition 6.10.** Let  $\hat{\eta} = \hat{\eta}^+ - \hat{\eta}^-$  be a real finite dimensional  $\mathbf{Z}_2$ -representation of  $K$  on  $E_{\hat{\eta}} = E_{\hat{\eta}^+} - E_{\hat{\eta}^-}$  such that the following identity in  $R(K_M, \mathbf{R})$  holds:

$$(6.62) \quad E_{\hat{\eta}}|_{K_M} = \sum_{i=0}^{\dim \mathfrak{p}_{\mathfrak{m}}} (-1)^i \Lambda^i(\mathfrak{p}_{\mathfrak{m}}^*) \otimes E_{\eta}|_{K_M}.$$

By [Sh16b, Section 5.3], for  $0 \leq j \leq 2l$ , the adjoint representation  $\eta_j$  of  $M$  on  $\Lambda^j(\mathfrak{n}^*)$  satisfies Assumption 6.9, such that the following identity in  $R(K, \mathbf{R})$  holds,

$$(6.63) \quad \sum_{i=1}^m (-1)^{i-1} i \Lambda^i(\mathfrak{p}^*) = \sum_{j=0}^{2l_0} (-1)^j E_{\hat{\eta}_j}.$$

**6.9. Evaluation of  $\text{Tr}_s^{[\gamma]}[\exp(-tC^{\mathfrak{g}, X, \hat{\eta}}/2)]$ .** In [Sh16b, Sections 5.4 and 5.5], we evaluate the orbital integral  $\text{Tr}_s^{[\gamma]}[\exp(-tC^{\mathfrak{g}, X, \hat{\eta}}/2)]$  when  $\gamma = 1$  or when  $\gamma$  is semisimple and non elliptic. In this subsection, we evaluate  $\text{Tr}_s^{[\gamma]}[\exp(-tC^{\mathfrak{g}, X, \hat{\eta}}/2)]$  when  $\gamma$  is elliptic. To state the result, let us introduce some notations.

Recall that  $T$  is a maximal torus of  $K_M$ ,  $K$  and  $U_M$ . Denote by  $W(T, U_M)$  and  $W(T, K)$  the corresponding Weyl groups, and by  $\text{vol}(K/K_M)$  and  $\text{vol}(U_M/K_M)$  the Riemannian volumes induced by  $-B$ . Set

$$(6.64) \quad c_G = (-1)^{(\dim \mathfrak{p}-1)/2} \frac{|W(T, U_M)|}{|W(T, K)|} \frac{\text{vol}(K/K_M)}{\text{vol}(U_M/K_M)} \in \mathbf{R}.$$

Take  $k \in T$ . We have  $\delta(Z^0(k)) = 1$ . Denote by  $c_{Z^0(k)}$ ,  $\mathfrak{n}(k)$ ,  $U^0(k)$ ,  $Y_{\mathfrak{b}}(k)$  and  $\omega^{Y_{\mathfrak{b}}(k)}$  the analogies of  $\mathfrak{n}$ ,  $U$ ,  $Y_{\mathfrak{b}}$  and  $\omega^{Y_{\mathfrak{b}}(k)}$  when  $G$  is replaced by  $Z^0(k)$ . The embedding  $U^0(k) \rightarrow U$  induces an embedding  $Y_{\mathfrak{b}}(k) \rightarrow Y_{\mathfrak{b}}$ . Clearly,  $k$  acts on the left on  $Y_{\mathfrak{b}}$ , and  $Y_{\mathfrak{b}}(k)$  is fixed by the action of  $k$ . Recall that the equivariant  $\hat{A}$ -forms  $\hat{A}_{k-1} \left( N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}, \nabla^{N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}} \right)$  and



$\widehat{A}_{k-1} \left( TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}, \nabla^{TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}} \right)$  are defined in (1.18). Let  $\widehat{A}_{k-1}^{\mathfrak{u}_{\mathfrak{m}}}(0)$  and  $\widehat{A}_{k-1}^{\mathfrak{u}^\perp(\mathfrak{b})}(0)$  be the components of degree 0 of  $\widehat{A}_{k-1} \left( N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}, \nabla^{N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}} \right)$  and  $\widehat{A}_{k-1} \left( TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}, \nabla^{TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}} \right)$ . Following [B11, (7.7.3)], we set

$$(6.65) \quad \widehat{A}_{k-1}(0) = \widehat{A}_{k-1}^{\mathfrak{u}_{\mathfrak{m}}}(0) \widehat{A}_{k-1}^{\mathfrak{u}^\perp(\mathfrak{b})}(0).$$

By (6.61), as in [B11, (7.7.5)], the following identity of closed forms on  $Y_{\mathfrak{b}}(k)$  holds:

$$(6.66) \quad \widehat{A}_{k-1}(0) = \widehat{A}_{k-1} \left( N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}, \nabla^{N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}} \right) \widehat{A}_{k-1} \left( TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}, \nabla^{TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}} \right).$$

Note that the Kähler form  $\omega^{Y_{\mathfrak{b}}(k)}$  defines a volume form  $dv_{Y_{\mathfrak{b}}(k)}$  on  $Y_{\mathfrak{b}}(k)$ . For a  $U^0(k)$ -invariant differential form  $\beta$  on  $Y_{\mathfrak{b}}(k)$ , as in [Sh16b, (7.131)], define  $[\beta]^{\max} \in \mathbf{R}$  such that

$$(6.67) \quad \beta - [\beta]^{\max} dv_{Y_{\mathfrak{b}}(k)}$$

has degree smaller than  $\dim Y_{\mathfrak{b}}(k)$ . Recall that  $a_0 \in \mathfrak{b}$  is defined in (6.55).

**Theorem 6.11.** *Let  $\gamma \in G$  be semisimple. If  $\gamma$  can not be conjugated into  $H$  by elements of  $G$ , then for  $t > 0$*

$$(6.68) \quad \mathrm{Tr}_s^{[\gamma]} \left[ \exp \left( -tC^{\mathfrak{g}, X, \widehat{\eta}}/2 \right) \right] = 0.$$

If  $\gamma = k^{-1} \in T$ , then for  $t > 0$ ,

$$(6.69) \quad \mathrm{Tr}^{[\gamma]} \left[ \exp \left( -tC^{\mathfrak{g}, X, \widehat{\eta}}/2 \right) \right] = \frac{cZ^0(k)}{\sqrt{2\pi t}} \exp \left( \frac{t}{16} \mathrm{Tr} \left[ C^{\mathfrak{u}(\mathfrak{b}), \mathfrak{u}^\perp(\mathfrak{b})} \right] - \frac{t}{2} C^{\mathfrak{u}_{\mathfrak{m}}, \eta} \right) \left[ \exp \left( -\frac{\omega^{Y_{\mathfrak{b}}(k), 2}}{8\pi^2 |a_0|^2 t} \right) \widehat{A}_{k-1} \left( TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}, \nabla^{TY_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}} \right) \mathrm{ch}_{k-1} \left( F_{\mathfrak{b}, \eta}|_{Y_{\mathfrak{b}}(k)}, \nabla^{F_{\mathfrak{b}, \eta}|_{Y_{\mathfrak{b}}(k)}} \right) \right]^{\max}.$$

If  $\gamma = e^a k^{-1} \in H$  with  $a \neq 0$ , for any  $t > 0$ ,

$$(6.70) \quad \mathrm{Tr}_s^{[\gamma]} \left[ \exp \left( -tC^{\mathfrak{g}, X, \widehat{\eta}}/2 \right) \right] = \frac{1}{\sqrt{2\pi t}} \left[ e \left( TX^{a, \perp}(\gamma), \nabla^{TX^{a, \perp}(\gamma)} \right) \right]^{\max} \exp \left( -\frac{|a|^2}{2t} + \frac{t}{16} \mathrm{Tr} \left[ C^{\mathfrak{u}(\mathfrak{b}), \mathfrak{u}^\perp(\mathfrak{b})} \right] - \frac{t}{2} C^{\mathfrak{u}_{\mathfrak{m}}, \eta} \right) \frac{\mathrm{Tr}^{E_\eta} [\eta(k^{-1})]}{\left| \det (1 - \mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^\perp} \right|^{1/2}}.$$

*Proof.* Equations (6.68), (6.69) with  $\gamma = 1$ , and (6.70) are [Sh16b, Theorem 5.15]. The proof of (6.69) for non trivial  $\gamma = k^{-1} \in T$  is similar. Let us begin with introducing some notation.

Take  $\gamma = k^{-1} \in T$ . Then,  $k$  acts on  $\mathfrak{p}_{\mathfrak{m}}$ ,  $\mathfrak{k}_{\mathfrak{m}}$ . Set

$$(6.71) \quad \begin{aligned} \mathfrak{p}_{\mathfrak{m}}(k) &= \mathfrak{p}_{\mathfrak{m}} \cap \mathfrak{z}(k), & \mathfrak{k}_{\mathfrak{m}}(k) &= \mathfrak{k}_{\mathfrak{m}} \cap \mathfrak{z}(k), & \mathfrak{m}(k) &= \mathfrak{m} \cap \mathfrak{z}(k), \\ \mathfrak{p}_{\mathfrak{m}}^\perp(k) &= \mathfrak{p}_{\mathfrak{m}} \cap \mathfrak{z}^\perp(k), & \mathfrak{k}_{\mathfrak{m}}^\perp(k) &= \mathfrak{k}_{\mathfrak{m}} \cap \mathfrak{z}^\perp(k), & \mathfrak{m}^\perp(k) &= \mathfrak{m} \cap \mathfrak{z}^\perp(k). \end{aligned}$$

Clearly,

$$(6.72) \quad \mathfrak{p}_{\mathfrak{m}} = \mathfrak{p}_{\mathfrak{m}}(k) \oplus \mathfrak{p}_{\mathfrak{m}}^\perp(k), \quad \mathfrak{k}_{\mathfrak{m}} = \mathfrak{k}_{\mathfrak{m}}(k) \oplus \mathfrak{k}_{\mathfrak{m}}^\perp(k).$$

Similarly,  $k$  acts on  $\mathfrak{p}^\perp(\mathfrak{b})$  and  $\mathfrak{k}^\perp(\mathfrak{b})$ . Set

$$(6.73) \quad \begin{aligned} \mathfrak{p}_1^\perp(\mathfrak{b}) &= \mathfrak{p}^\perp(\mathfrak{b}) \cap \mathfrak{z}(k), & \mathfrak{k}_1^\perp(\mathfrak{b}) &= \mathfrak{k}^\perp(\mathfrak{b}) \cap \mathfrak{z}(k), & \mathfrak{z}_1^\perp(k) &= \mathfrak{z}^\perp(\mathfrak{b}) \cap \mathfrak{z}(k), \\ \mathfrak{p}_2^\perp(\mathfrak{b}) &= \mathfrak{p}^\perp(\mathfrak{b}) \cap \mathfrak{z}^\perp(k), & \mathfrak{k}_2^\perp(\mathfrak{b}) &= \mathfrak{k}^\perp(\mathfrak{b}) \cap \mathfrak{z}^\perp(k), & \mathfrak{z}_2^\perp(k) &= \mathfrak{z}^\perp(\mathfrak{b}) \cap \mathfrak{z}^\perp(k). \end{aligned}$$

Then

$$(6.74) \quad \mathfrak{p}^\perp(\mathfrak{b}) = \mathfrak{p}_1^\perp(\mathfrak{b}) \oplus \mathfrak{p}_2^\perp(\mathfrak{b}), \quad \mathfrak{k}^\perp(\mathfrak{b}) = \mathfrak{k}_1^\perp(\mathfrak{b}) \oplus \mathfrak{k}_2^\perp(\mathfrak{b}).$$

By (6.71) and (6.73), we get

$$(6.75) \quad \begin{aligned} \mathfrak{p}(k) &= \mathfrak{b} \oplus \mathfrak{p}_m(k) \oplus \mathfrak{p}_1^\perp(\mathfrak{b}), & \mathfrak{k}(k) &= \mathfrak{k}_m(k) \oplus \mathfrak{k}_1^\perp(\mathfrak{b}), \\ \mathfrak{p}^\perp(k) &= \mathfrak{p}_m^\perp(k) \oplus \mathfrak{p}_2^\perp(\mathfrak{b}), & \mathfrak{k}^\perp(k) &= \mathfrak{k}_m^\perp(k) \oplus \mathfrak{k}_2^\perp(\mathfrak{b}). \end{aligned}$$

As in the case of [Sh16b, (5.9)], we have isomorphisms of representations of  $T$ ,

$$(6.76) \quad \mathfrak{p}_1^\perp(\mathfrak{b}) \simeq \mathfrak{k}_1^\perp(\mathfrak{b}) \simeq \mathfrak{n}(k),$$

where the first isomorphism is given by  $\text{ad}(a_0)$ . Moreover,  $\text{ad}(a_0)$  induces an isomorphism of representations of  $T$ ,

$$(6.77) \quad \mathfrak{p}_2^\perp(\mathfrak{b}) \simeq \mathfrak{k}_2^\perp(\mathfrak{b}).$$

Set

$$(6.78) \quad \begin{aligned} \mathfrak{u}_m(k) &= \sqrt{-1}\mathfrak{p}_m(k) \oplus \mathfrak{k}_m(k), & \mathfrak{u}_m^\perp(k) &= \sqrt{-1}\mathfrak{p}_m^\perp(k) \oplus \mathfrak{k}_m^\perp(k), \\ \mathfrak{u}_1^\perp(\mathfrak{b}) &= \sqrt{-1}\mathfrak{p}_1^\perp(\mathfrak{b}) \oplus \mathfrak{k}_1^\perp(\mathfrak{b}), & \mathfrak{u}_2^\perp(\mathfrak{b}) &= \sqrt{-1}\mathfrak{p}_2^\perp(\mathfrak{b}) \oplus \mathfrak{k}_2^\perp(\mathfrak{b}). \end{aligned}$$

By (6.29) and by the Weyl integral formula for Lie algebra [Sh16b, (5.82)], we have

$$(6.79) \quad \begin{aligned} &\text{Tr}_s^{[\gamma]} [\exp(-tC^{\mathfrak{g},X,\hat{\eta}}/2)] \\ &= \frac{1}{(2\pi t)^{\dim \mathfrak{z}(k)/2}} \exp\left(\frac{t}{16} \text{Tr}^{\mathfrak{p}} [C^{\mathfrak{k},\mathfrak{p}}] + \frac{t}{48} \text{Tr}^{\mathfrak{k}} [C^{\mathfrak{k},\mathfrak{k}}]\right) \frac{\text{vol}(K^0(k)/T)}{|W(T, K^0(k))|} \\ &\quad \int_{Y \in \mathfrak{t}} \det(\text{ad}(Y))|_{\mathfrak{k}(k)/\mathfrak{t}} J_{k^{-1}}(Y) \text{Tr}_s^{E_{\hat{\eta}}} [\hat{\eta}(k^{-1}) \exp(-i\hat{\eta}(Y))] \exp(-|Y|^2/2t) dY. \end{aligned}$$

As  $\mathfrak{k}$  is also the Cartan subalgebra of  $\mathfrak{u}_m(k)$ , we will rewrite the integral on the right-hand side as an integral over  $\mathfrak{u}_m(k)$  as [Sh16b, Theorem 5.15].

By (6.26), (6.75)-(6.77), for  $Y \in \mathfrak{t}$ , we have

$$(6.80) \quad J_{k^{-1}}(Y) = \frac{\hat{A}(i \text{ad}(Y)|_{\mathfrak{p}_m(k)})}{\hat{A}(i \text{ad}(Y)|_{\mathfrak{k}_m(k)})} \left[ \det(1 - \text{Ad}(k^{-1}))|_{\mathfrak{z}_2^\perp(\mathfrak{b})} \right]^{-1/2} \left[ \frac{1}{\det(1 - \text{Ad}(k^{-1}))|_{\mathfrak{z}_m(k)}} \frac{\det(1 - \exp(-i \text{ad}(Y)) \text{Ad}(k^{-1}))|_{\mathfrak{k}_m(k)}}{\det(1 - \exp(-i \text{ad}(Y)) \text{Ad}(k^{-1}))|_{\mathfrak{p}_m(k)}} \right]^{1/2}.$$

By (6.62), (6.76) and (6.80), for  $Y \in \mathfrak{t}$ , we have

$$(6.81) \quad \begin{aligned} &\frac{\det(\text{ad}(Y))|_{\mathfrak{k}(k)/\mathfrak{t}}}{\det(\text{ad}(Y))|_{\mathfrak{u}_m(k)/\mathfrak{t}}} J_{k^{-1}}(Y) \text{Tr}_s^{E_{\hat{\eta}}} [\hat{\eta}(k^{-1}) \exp(-i\hat{\eta}(Y))] = (-1)^{\frac{\dim \mathfrak{p}_m(k)}{2}} \\ &\quad \det(\text{ad}(Y))|_{\mathfrak{n}(k)} \hat{A}^{-1}(i \text{ad}(Y)|_{\mathfrak{u}_m(k)}) \text{Tr}_s^{E_{\eta}} [\eta(k^{-1}) \exp(-i\eta(Y))] \\ &\quad \left[ \det(1 - \text{Ad}(k^{-1}))|_{\mathfrak{u}_2^\perp(\mathfrak{b})} \right]^{-1/2} \left[ \frac{\det(1 - \exp(-i \text{ad}(Y)) \text{Ad}(k^{-1}))|_{\mathfrak{u}_m^\perp(k^{-1})}}{\det(1 - \text{Ad}(k^{-1}))|_{\mathfrak{u}_m^\perp(k^{-1})}} \right]^{1/2}. \end{aligned}$$

Let  $U_M(k)$  be the centralizer of  $k$  in  $U_M$ , and let  $U_M^0(k)$  be the connected component of the identity in  $U_M(k)$ . The right-hand side of (6.81) is  $\text{Ad}(U_M^0(k))$ -invariant. By (6.64),

(6.79) and (6.81), and using again the Weyl integral formula, we get

$$(6.82) \quad \mathrm{Tr}_s^{[\gamma]} \left[ \exp \left( -t C^{\mathfrak{g}, X, \hat{\eta}} / 2 \right) \right] = \frac{(-1)^{\frac{\dim \mathfrak{n}(k)}{2}} \exp \left( \frac{t}{16} \mathrm{Tr}^{\mathfrak{p}} [C^{\mathfrak{k}, \mathfrak{p}}] + \frac{t}{48} \mathrm{Tr}^{\mathfrak{k}} [C^{\mathfrak{k}, \mathfrak{k}}] \right)}{(2\pi t)^{\dim \mathfrak{z}(k)/2} \left[ \det (1 - \mathrm{Ad}(k^{-1}))|_{\mathfrak{u}_2^\perp(\mathfrak{b})} \right]^{1/2}} \\ c_{Z^0(k)} \int_{Y \in \mathfrak{u}_{\mathfrak{m}}(k)} \det (\mathrm{ad}(Y))|_{\mathfrak{n}(k)} \hat{A}^{-1} (i \mathrm{ad}(Y)|_{\mathfrak{u}_{\mathfrak{m}}(k)}) \mathrm{Tr}^{E_\eta} [\eta(k^{-1}) \exp(-i\eta(Y))] \\ \left[ \frac{\det (1 - \exp(-i \mathrm{ad}(Y)) \mathrm{Ad}(k^{-1}))|_{\mathfrak{u}_{\mathfrak{m}}^\perp(k)}}{\det (1 - \mathrm{Ad}(k^{-1}))|_{\mathfrak{u}_{\mathfrak{m}}^\perp(k)}} \right]^{1/2} \exp(-|Y|^2/2t) dY.$$

Proceeding as in [Sh16b, (5.90)-(5.105), (5.107)-(5.109)], we get

$$(6.83) \quad \mathrm{Tr}_s^{[\gamma]} \left[ \exp \left( -t C^{\mathfrak{g}, X, \hat{\eta}} / 2 \right) \right] = \frac{c_{Z^0(k)}}{\sqrt{2\pi t}} \exp \left( \frac{t}{16} \mathrm{Tr} \left[ C^{\mathfrak{u}(\mathfrak{b}), \mathfrak{u}^\perp(\mathfrak{b})} \right] - \frac{t}{2} C^{\mathfrak{u}_{\mathfrak{m}}, \eta} \right) \\ \left[ \exp \left( -\frac{\omega^{Y_{\mathfrak{b}}(k), 2}}{8\pi^2 |a_0|^2 t} \right) \hat{A}_{k^{-1}}^{-1}(0) \hat{A}_{k^{-1}}^{-1} \left( N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}, \nabla^{N_{\mathfrak{b}}|_{Y_{\mathfrak{b}}(k)}} \right) \mathrm{ch}_{k^{-1}} \left( F_{\mathfrak{b}, \eta}|_{Y_{\mathfrak{b}}(k)}, \nabla^{F_{\mathfrak{b}, \eta}|_{Y_{\mathfrak{b}}(k)}} \right) \right]^{\max}.$$

By (6.66) and (6.83), we get (6.69). The proof of Theorem 6.11 is completed.  $\square$

**6.10. Selberg-type zeta functions.** In this subsection, we introduce a Selberg-type zeta function associated to the representation  $\eta$  of  $M$  satisfying Assumption 6.9. By (6.37), by Proposition 6.4, Theorem 6.11, proceeding as in [Sh16b, Proposition 7.27], we find that there is  $\sigma_0 > 0$  such that

$$(6.84) \quad \sum_{[\gamma] \in [\Gamma_+], \gamma = g_\gamma e^a k^{-1} g_\gamma^{-1}} \frac{|\chi_{\mathrm{orb}}(\mathbb{S}^1 \setminus B_{[\gamma]})|}{m_{[\gamma]}} \frac{|a| e^{-\sigma_0 |a|}}{\left| \det (1 - \mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^\perp} \right|^{1/2}} < \infty.$$

Note that there is  $c_0 > 0$  such that for  $\gamma \in \Gamma_+$  with  $\gamma = g_\gamma e^a k^{-1} g_\gamma^{-1}$  as in (6.13), we have

$$(6.85) \quad |a| \geq c_0.$$

By (6.84) and (6.85), for  $\sigma \in \mathbb{C}$  and  $\mathrm{Re}(\sigma) > \sigma_0$ , the sum

$$(6.86) \quad \Xi_{\eta, \rho}(\sigma) = - \sum_{[\gamma] \in [\Gamma_+], \gamma = g_\gamma e^a k^{-1} g_\gamma^{-1}} \mathrm{Tr}[\rho(\gamma)] \frac{\chi_{\mathrm{orb}}(\mathbb{S}^1 \setminus B_{[\gamma]})}{m_{[\gamma]}} \frac{\mathrm{Tr}^{E_\eta} [\eta(k^{-1})]}{\left| \det (1 - \mathrm{Ad}(\gamma))|_{\mathfrak{z}_0^\perp} \right|^{1/2}} e^{-\sigma |a|}$$

converges absolutely to a holomorphic function defined on this domain.

**Definition 6.12.** For  $\sigma \in \mathbb{C}$  and  $\mathrm{Re}(\sigma) > \sigma_0$ , we define a Selberg-type zeta function by

$$(6.87) \quad Z_{\eta, \rho}(\sigma) = \exp (\Xi_{\eta, \rho}(\sigma)).$$

Recall that the Casimir operator  $C^{g, Z, \hat{\eta}, \rho}$  acting on  $C^\infty(Z, \mathcal{F}_{\hat{\eta} \otimes_{\mathbb{C}} F})$  is a formally self-adjoint second order elliptic operator, which is bounded from below. Set

$$(6.88) \quad m_{\eta, \rho}(\lambda) = \dim_{\mathbb{C}} \ker (C^{g, Z, \hat{\eta}^+, \rho} - \lambda) - \dim_{\mathbb{C}} \ker (C^{g, Z, \hat{\eta}^-, \rho} - \lambda).$$

As in [Sh16b, (5.130)], consider the quotient of zeta regularized determinants

$$(6.89) \quad \det_{\mathrm{gr}} (C^{g, Z, \hat{\eta}, \rho} + \sigma) = \frac{\det (C^{g, Z, \hat{\eta}^+, \rho} + \sigma)}{\det (C^{g, Z, \hat{\eta}^-, \rho} + \sigma)}.$$

Thus, it is a meromorphic function on  $\mathbb{C}$ . Its zeros and poles belong to the set  $\{-\lambda : \lambda \in \mathrm{Sp}(C^{g, Z, \hat{\eta}, \rho})\}$ . The order of zero at  $\sigma = -\lambda$  is  $m_{\eta, \rho}(\lambda)$ .

Set

$$(6.90) \quad \sigma_\eta = \frac{1}{8} \operatorname{Tr} \left[ C^{\mathfrak{u}(\mathfrak{b}), \mathfrak{u}^\perp(\mathfrak{b})} \right] - C^{\mathfrak{u}_\mathfrak{m}, \eta}.$$

Let  $P_{\eta, \rho}(\sigma)$  be the odd polynomial defined by

$$(6.91) \quad \begin{aligned} P_{\eta, \rho}(\sigma) = & \sum_{[\gamma] \in [\Gamma_e], \gamma = g_\gamma k^{-1} g_\gamma^{-1}} c_{Z^0(k)} \operatorname{Tr}[\rho(\gamma)] \frac{\operatorname{vol}(\Gamma(\gamma) \backslash X(\gamma))}{|\delta(\gamma)|} \left( \sum_{j=0}^{\dim \mathfrak{n}(k)/2} (-1)^j \frac{\Gamma(-j - \frac{1}{2})}{j!(4\pi)^{2j + \frac{1}{2}} |a_0|^{2j}} \right. \\ & \times \left. \left[ \omega^{Y_\mathfrak{b}(k), 2j} \widehat{A}_{k-1} \left( TY_\mathfrak{b}|_{Y_\mathfrak{b}(k)}, \nabla^{TY_\mathfrak{b}|_{Y_\mathfrak{b}(k)}} \right) \operatorname{ch}_{k-1} \left( \mathcal{F}_{\mathfrak{b}, \eta}|_{Y_\mathfrak{b}(k)}, \nabla^{\mathcal{F}_{\mathfrak{b}, \eta}|_{Y_\mathfrak{b}(k)}} \right) \right]^{\max} \sigma^{2j+1} \right). \end{aligned}$$

**Theorem 6.13.** *The Selberg-type zeta function  $Z_{\eta, \rho}(\sigma)$  has a meromorphic extension to  $\sigma \in \mathbb{C}$  such that the following identity of meromorphic functions on  $\mathbb{C}$  holds:*

$$(6.92) \quad Z_{\eta, \rho}(\sigma) = \det_{\text{gr}} \left( C^{\mathfrak{g}, Z, \widehat{\eta}, \rho} + \sigma_\eta + \sigma^2 \right) \exp \left( P_{\eta, \rho}(\sigma) \right).$$

*The zeros and poles of  $Z_{\eta, \rho}(\sigma)$  belong to the set  $\{\pm i \sqrt{\lambda + \sigma_\eta} : \lambda \in \operatorname{Sp}(C^{\mathfrak{g}, Z, \widehat{\eta}, \rho})\}$ . If  $\lambda \in \operatorname{Sp}(C^{\mathfrak{g}, Z, \widehat{\eta}, \rho})$  and  $\lambda \neq -\sigma_\eta$ , the order of zero at  $\sigma = \pm i \sqrt{\lambda + \sigma_\eta}$  is  $m_{\eta, \rho}(\lambda)$ . The order of zero at  $\sigma = 0$  is  $2m_{\eta, \rho}(-\sigma_\eta)$ . Also,*

$$(6.93) \quad Z_{\eta, \rho}(\sigma) = Z_{\eta, \rho}(-\sigma) \exp \left( P_{\eta, \rho}(\sigma) \right).$$

*Proof.* Proceeding as in [Sh16b, Theorem 5.21], by Theorem 6.11 and (6.92), Theorem 6.13 follows.  $\square$

**6.11. The proof of Theorem 6.6 when  $\delta(G) = 1$ .** We apply the results of subsection 6.10 to  $\eta_j$ . Recall that  $\alpha \in \mathfrak{b}^*$  is defined in (6.55). Proceeding as in [Sh16b, Theorem 5.22], by (6.63), we find that  $R_\rho(\sigma)$  is well-defined and holomorphic on the domain  $\sigma \in \mathbb{C}$  and  $\operatorname{Re}(\sigma) \gg 1$ , and that

$$(6.94) \quad R_\rho(\sigma) = \prod_{j=0}^{2l} Z_{\eta_j, \rho}(\sigma + (j-l)|\alpha|)^{(-1)^{j-1}}.$$

By Theorem 6.13 and (6.94),  $R_\rho(\sigma)$  has a meromorphic extension to  $\sigma \in \mathbb{C}$ .

For  $0 \leq j \leq 2l$ , put

$$(6.95) \quad r_j = m_{\eta_j, \rho}(0).$$

As in [Sh16b, 5.(157)], we have

$$(6.96) \quad \chi'_{\text{top}}(Z, F) = 2 \sum_{j=0}^{l-1} (-1)^{j-1} r_j + (-1)^{l-1} r_l.$$

Set

$$(6.97) \quad C_\rho = \prod_{j=0}^{l-1} \left( -4(l-j)^2 |\alpha|^2 \right)^{(-1)^{j-1} r_j}, \quad r_\rho = 2 \sum_{j=0}^l (-1)^{j-1} r_j.$$

Note that if  $G$  has noncompact center, then  $l_0 = 0$  and

$$(6.98) \quad C_\rho = 1, \quad r_\rho = -2r_0.$$

Proceeding as in [Sh16b, Section 5.7], we get (6.45). As [Sh16b, Remark 5.26], we get (6.46) when  $G$  has a noncompact center. If  $G$  has a compact center, proceeding as [Sh16b, Section 7], for all  $0 \leq j \leq 2l$ , we have

$$(6.99) \quad r_j = 0.$$

By (6.97), we get (6.46). The proof of Theorem 6.6 is completed.  $\square$

## REFERENCES

- [AdLeRu07] A. Adem, J. Leida, and Y. Ruan, *Orbifolds and stringy topology*, Cambridge Tracts in Mathematics, vol. 171, Cambridge University Press, Cambridge, 2007. MR 2359514
- [ABo67] M. F. Atiyah and R. Bott, *A Lefschetz fixed point formula for elliptic complexes. I*, Ann. of Math. (2) **86** (1967), 374–407. MR 0212836
- [ABo68] ———, *A Lefschetz fixed point formula for elliptic complexes. II. Applications*, Ann. of Math. (2) **88** (1968), 451–491. MR 0232406
- [BeGeVe04] N. Berline, E. Getzler, and M. Vergne, *Heat kernels and Dirac operators*, Grundlehren Text Editions, Springer-Verlag, Berlin, 2004, Corrected reprint of the 1992 original. MR 2273508 (2007m:58033)
- [B05] J.-M. Bismut, *The hypoelliptic Laplacian on the cotangent bundle*, J. Amer. Math. Soc. **18** (2005), no. 2, 379–476 (electronic). MR 2137981 (2006f:35036)
- [B11] ———, *Hypoelliptic Laplacian and orbital integrals*, Annals of Mathematics Studies, vol. 177, Princeton University Press, Princeton, NJ, 2011. MR 2828080
- [BG01] J.-M. Bismut and S. Goette, *Families torsion and Morse functions*, Astérisque (2001), no. 275, x+293. MR 1867006 (2002h:58059)
- [BG04] ———, *Equivariant de Rham torsions*, Ann. of Math. (2) **159** (2004), no. 1, 53–216. MR 2051391 (2005f:58059)
- [BL95] J.-M. Bismut and J. Lott, *Flat vector bundles, direct images and higher real analytic torsion*, J. Amer. Math. Soc. **8** (1995), no. 2, 291–363. MR 1303026 (96g:58202)
- [BZ92] J.-M. Bismut and W. Zhang, *An extension of a theorem by Cheeger and Müller*, Astérisque (1992), no. 205, 235, With an appendix by François Laudenbach. MR 1185803 (93j:58138)
- [BZ94] ———, *Milnor and Ray-Singer metrics on the equivariant determinant of a flat vector bundle*, Geom. Funct. Anal. **4** (1994), no. 2, 136–212. MR 1262703 (96f:58179)
- [BrH99] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 1744486
- [C79] J. Cheeger, *Analytic torsion and the heat equation*, Ann. of Math. (2) **109** (1979), no. 2, 259–322. MR 528965 (80j:58065a)
- [ChP81] J. Chazarain and A. Piriou, *Introduction à la théorie des équations aux dérivées partielles linéaires*, Gauthier-Villars, Paris, 1981. MR 598467
- [DY16] X. Dai and J. Yu, *Comparison between two analytic torsions on orbifolds*, Mathematische Zeitschrift (2016), 1–14.
- [DuKoV79] J. J. Duistermaat, J. A. C. Kolk, and V. S. Varadarajan, *Spectra of compact locally symmetric manifolds of negative curvature*, Invent. Math. **52** (1979), no. 1, 27–93. MR 532745 (82a:58050a)
- [DyZ16a] S. Dyatlov and M. Zworski, *Dynamical zeta functions for Anosov flows via microlocal analysis*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 3, 543–577. MR 3503826
- [DyZ16b] ———, *Ruelle zeta function at zero for surfaces*, arXiv:1606.04560 (2016).
- [Fa07] C. Farsi, *Orbifold  $\eta$ -invariants*, Indiana Univ. Math. J. **56** (2007), no. 2, 501–521. MR 2317536
- [Fe15a] K. Fedosova, *The twisted selberg trace formula and the selberg zeta function for compact orbifolds*, arXiv:1511.04208 (2015).
- [Fe15b] ———, *On the asymptotics of the analytic torsion for compact hyperbolic orbifolds*, arXiv:1511.04281 (2015).
- [Fe16] ———, *Analytic torsion of finite volume hyperbolic orbifolds*, arXiv:1601.07873 (2016).

- [F86] D. Fried, *Analytic torsion and closed geodesics on hyperbolic manifolds*, Invent. Math. **84** (1986), no. 3, 523–540. MR 837526 (87g:58118)
- [F87] ———, *Lefschetz formulas for flows*, The Lefschetz centennial conference, Part III (Mexico City, 1984), Contemp. Math., vol. 58, Amer. Math. Soc., Providence, RI, 1987, pp. 19–69. MR 893856 (88k:58138)
- [GiLiPo13] P. Giulietti, C. Liverani, and M. Pollicott, *Anosov flows and dynamical zeta functions*, Ann. of Math. (2) **178** (2013), no. 2, 687–773. MR 3071508
- [H90] A. Haefliger, *Orbi-espaces*, Sur les groupes hyperboliques d’après Mikhael Gromov (Bern, 1988), Progr. Math., vol. 83, Birkhäuser Boston, Boston, MA, 1990, pp. 203–213. MR 1086659
- [K78] T. Kawasaki, *The signature theorem for V-manifolds*, Topology **17** (1978), no. 1, 75–83. MR 0474432
- [K79] ———, *The Riemann-Roch theorem for complex V-manifolds*, Osaka J. Math. **16** (1979), no. 1, 151–159. MR 527023
- [Kn86] A. W. Knap, *Representation theory of semisimple groups*, Princeton Mathematical Series, vol. 36, Princeton University Press, Princeton, NJ, 1986, An overview based on examples. MR 855239 (87j:22022)
- [Kn02] ———, *Lie groups beyond an introduction*, second ed., Progress in Mathematics, vol. 140, Birkhäuser Boston, Inc., Boston, MA, 2002. MR 1920389 (2003c:22001)
- [Ma05] X. Ma, *Orbifolds and analytic torsions*, Trans. Amer. Math. Soc. **357** (2005), no. 6, 2205–2233 (electronic). MR 2140438
- [MaMar07] X. Ma and G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, Progress in Mathematics, vol. 254, Birkhäuser Verlag, Basel, 2007. MR 2339952 (2008g:32030)
- [M68] J. Milnor, *Infinite cyclic coverings*, Conference on the Topology of Manifolds (Michigan State Univ., E. Lansing, Mich., 1967), Prindle, Weber & Schmidt, Boston, Mass., 1968, pp. 115–133. MR 0242163 (39 #3497)
- [Mos70] G. D. Mostow, *Intersections of discrete subgroups with Cartan subgroups*, J. Indian Math. Soc. **34** (1970), no. 3-4, 203–214 (1971). MR 0492074 (58 #11228)
- [MoePr97] I. Moerdijk and D. A. Pronk, *Orbifolds, sheaves and groupoids*, K-Theory **12** (1997), no. 1, 3–21. MR 1466622
- [McS67] H. P. McKean and I. M. Singer, *Curvature and the eigenvalues of the Laplacian*, J. Differential Geometry **1** (1967), no. 1, 43–69. MR 0217739 (36 #828)
- [MoSt91] H. Moscovici and R. J. Stanton, *R-torsion and zeta functions for locally symmetric manifolds*, Invent. Math. **105** (1991), no. 1, 185–216. MR 1109626 (92i:58199)
- [M78] W. Müller, *Analytic torsion and R-torsion of Riemannian manifolds*, Adv. in Math. **28** (1978), no. 3, 233–305. MR 498252 (80j:58065b)
- [M93] ———, *Analytic torsion and R-torsion for unimodular representations*, J. Amer. Math. Soc. **6** (1993), no. 3, 721–753. MR 1189689 (93m:58119)
- [Q85] D. Quillen, *Superconnections and the Chern character*, Topology **24** (1985), no. 1, 89–95. MR 790678 (86m:58010)
- [Re36] K. Reidemeister, *Kommutative Fundamentalgruppen*, Monatsh. Math. Phys. **43** (1936), no. 1, 20–28. MR 1550506
- [RS71] D. B. Ray and I. M. Singer, *R-torsion and the Laplacian on Riemannian manifolds*, Advances in Math. **7** (1971), 145–210. MR 0295381 (45 #4447)
- [Sa56] I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363. MR 0079769
- [Sa57] ———, *The Gauss-Bonnet theorem for V-manifolds*, J. Math. Soc. Japan **9** (1957), 464–492. MR 0095520 (20 #2022)
- [Se67] R. T. Seeley, *Complex powers of an elliptic operator*, Singular Integrals (Proc. Sympos. Pure Math., Chicago, Ill., 1966), Amer. Math. Soc., Providence, R.I., 1967, pp. 288–307. MR 0237943 (38 #6220)
- [Sh16a] S. Shen, *Analytic torsion, dynamical zeta functions and orbital integrals*, C. R. Math. Acad. Sci. Paris **354** (2016), no. 4, 433–436. MR 3473562



- [Sh16b] ———, *Analytic torsion, dynamical zeta functions and orbital integrals*, soumis, arXiv:1602.00664 (2016).
- [Sm61] S. Smale, *On gradient dynamical systems*, Ann. of Math. (2) **74** (1961), 199–206. MR 0133139 (24 #A2973)
- [Sm67] ———, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817. MR 0228014 (37 #3598)
- [T80] W. P. Thurston, *The geometry and topology of three-manifolds*, unpublished manuscript, <http://www.msri.org/publications/books/gt3m/>, 1980.
- [Vo87] A. Voros, *Spectral functions, special functions and the Selberg zeta function*, Comm. Math. Phys. **110** (1987), no. 3, 439–465. MR 891947

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